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E. T. BELL

F. D. MURNAGHAN

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ANNOUNCEMENT

An agreement has been effected between the Johns Hopkins University and the American Mathematical Society for coöperative publication of the AMERICAN JOURNAL OF MATHEMATICS. The Journal will be enlarged so as to help to meet the rapidly growing demands for the reasonably prompt publication of memoirs. There is no intention of changing the policy.

Substantial theories have, of course, the first place. As far as possible illustrations of such theories should be included, as they facilitate enormously the comprehension of the reader and thereby the spreading of the theories.

In the second place the Journal will welcome the converse process of elucidating a specific problem either with the aid of available theories or as pointing to further theories.

In the third place the Journal will gladly accept the applications of theories; i. e., the illumination of methods however general by specific and easily comprehended cases.

It is hoped that Applied Mathematics may be a valued feature in the next lap of the Journal's career.

G. D. BIRKHOFF,
A. B. COBLE,
ABRAHAM COHEN,
G. C. EVANS,
FRANK MORLEY, *Chairman.*

Stability and the Equations of Dynamics.*

By George D. Birkhoff.

Introduction.

For a long time attention has been directed towards the equations of dynamics, and more and more there has been the tendency to take these as being derivable from a special type of variational principle and as consequently possessing 'Hamiltonian' or 'canonical' form. In consequence of the fact that the mathematical treatment of these equations involves difficulties due to the lack of flexibility of the underlying group of transformations, a very large formal literature has grown up. On the other hand, physicists, realizing the sufficiency of such equations for most applications and impressed with the purely mathematical developments, have yielded a position of the very first importance to them, just as have the mathematicians because of the physical applications.

For these reasons there has been a disposition to attribute an almost mysterious significance to the Hamiltonian equations.

Now there is one property of these equations, first explicitly and adequately recognized by Poincaré, which is of obvious significance. If there is given an equilibrium position or a periodic motion of a dynamical system of this type, which satisfies the simplest necessary conditions for stability, and if there are no relations of commensurability between the underlying periods, then there will be complete formal trigonometric stability. This fact is of course of great physical interest, and throws great light upon the inward nature of the stability existing in the solar system for instance.

It is the main contention of this paper that essentially the *only* significance of the Hamiltonian (and thus the Lagrangian) equations is that they possess this characteristic property of complete formal stability. In other words, any set of n equations possessing complete formal stability at an equilibrium point, for example, may be given Hamiltonian form by appropriate formal change of variables.

In the present paper attention is primarily directed to the case of a

* See a preliminary note in the *Comptes rendus* for September 20, 1926. The present paper was read at the Philadelphia meeting of the American Mathematical Society on December 29, 1926.

system of equations of even order, although it is proved that the odd order case essentially reduces to that of even order. Furthermore, the treatment is restricted to the typical case of the equilibrium point, although I expect to treat the analogous important case of a periodic motion elsewhere, in so far as the generalization is not an obvious one.

The precise nature of the Hamiltonian equations is not adequately expressed in the mere statement that the property of stability is their one characteristic feature. Indeed, except from a purely formal point of view, these equations form a particular kind of completely stable systems, because of their convergence properties. The interrelation between the various kinds of stability of systems of ordinary differential equations, the nature of the trigonometric series employed and their degree of accuracy, the formal reduction to Hamiltonian or Pfaffian form in the case of complete stability, must be studied in detail before the very interesting situation which exists can be rightly comprehended. It is my purpose in this paper to clarify this situation as far as possible.

Many other problems are immediately suggested. I will only refer to one. In recent work on the theory of relativity there have been notable attempts made by Eddington, Einstein, Hilbert and Weyl to derive the general partial differential equations of physics from a 'Hamiltonian principle.' There arises now the question: Is the significance of the principle the same here as in classical dynamics, namely to insure the complete trigonometric stability of equilibrium or periodic motion?

In concluding these introductory remarks, I wish to express the hope that these important and interesting questions will receive the further attention which they certainly deserve. In the present paper only a beginning is made.

§ 1. *Definition of Stability.*

We propose to consider a system of n ordinary differential equations of the first order

$$(1) \qquad dx_i/dt = X_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

where the right-hand members are real analytic functions of $x_1, \cdots, x_n$ in the open connected continuum under consideration. In particular we shall study the properties of the solutions of (1) in the neighborhood of an 'equilibrium point' of this continuum, i. e., a point for which all of the functions

X_i vanish. It is especially questions concerning the stability of nearby motions that will occcupy our attention.

There are two quite distinct properties of stability of equilibrium: the first is that the nearby motions must continue near to equilibrium for a certain time; the second is that these nearby motions have a quasi-periodic character. We propose to incorporate both of these conditions in our definition of stability.

Definition of Stability. An equilibrium point of the differential system (1), taken at the origin in $x_1, \cdots, x_n$ space, will be said to be *stable of the mth order* if it possesses the two following properties: (A) all points lying at a distance ϵ from the origin at time t_0 remain at a distance not exceeding $K\epsilon$ for an interval $|t - t_0| \leqq L\epsilon^{-m}$ (K, L, definite positive constants); (B) for any fixed interval $|t - t_0| \leqq T$ the coordinates $x_1, \cdots, x_n$ of any such motion and polynomials with lowest terms of degree $s > 0$ in the coordinates can be represented by trigonometric sums

$$A_0 + \sum (A_i \cos p_i t + B_i \sin p_i t) \qquad (|p_i - p_j| \geqq P > 0)$$

of not more than $N + 1$ terms with an error not exceeding $Q\epsilon^{m+s}$ (P, N, Q, definite positive constants) during the fixed interval.

If the equilibrium point is stable of all orders it will be said to be *completely stable.*

It is obvious that stability of any order necessitates stability of all lower orders.

The equations (1) are invariant in form under any point transformation

$$x_i = \phi_i(\bar{x}_1, \cdots, \bar{x}_n) \qquad (i = 1, \cdots, n)$$

where the functions ϕ_i are analytic with $|\partial\phi_i/\partial\bar{x}_j| \neq 0$ at the transformed equilibrium point. The order of stability is clearly invariant under such a change of variables. In employing these transformations we shall always take the transformed point to be at the origin also, so that ϕ_i are given by power series in $\bar{x}_1, \cdots, \bar{x}_n$ without constant terms.

It will be one of our principal purposes to show the relation between these properties (A) and (B), and we shall find that 'perturbative stability' (A) does not necessitate 'trigonometric stability' (B), though it turns out that the requirement of trigonometric stability suffices to insure the stability of perturbations. We find it convenient first to go as far as possible on the basis of property (A) alone, and then employ the more fundamental requirement (B).

It is extremely instructive to consider a few characteristic examples in this connection.

The simplest possible example of a point of equilibrium is furnished by the equation

$$dx/dt = ax,$$

in which the point lies at $x = 0$. Here the exponential form of solution shows that there can not be first order stability even in the sense (A).

A more instructive example is furnished by the equations

$$dx/dt = y + \tfrac{1}{2}x(x^2 + y^2), \qquad dy/dt = -x + \tfrac{1}{2}y(x^2 + y^2)$$

with equilibrium point at (0, 0). The general solution is easily determined to be

$$x = \frac{r_0 \cos(\theta_0 + t - t_0)}{\sqrt{1 - r_0^2(t - t_0)}}, \quad y = \frac{r_0 \sin(\theta_0 + t - t_0)}{\sqrt{1 - r_0^2(t - t_0)}},$$

where r_0, θ_0 are the polar coordinates of (x, y) at t_0. Thus r_0 is the distance ϵ which enters into the above definition of stability.

In this case the expressions for x and y remain of the order of ϵ for $|t - t_0| \leqq L\epsilon^{-2}$ $(0 < L < 1)$, so that the equilibrium point is stable of the second order in the sense (A). But since the denominator in the expressions for x and y vanishes for $t - t_0 = \epsilon^{-2}$, it is obvious that the point is not stable of the third order. The approximate trigonometric form of the solutions with an error of order 3 in ϵ for any fixed interval $|t - t_0| \leqq T$ shows that the situation is the same for stability in the sense (B).

On the other hand, for the equations

$$dx/dt = y + y(x^2 + y^2), \qquad dy/dt = -x - x(x^2 + y^2),$$

the general solution is readily found to be

$$\begin{cases} x = x_0 \cos a(t - t_0) - y_0 \sin a(t - t_0) & (a = 1 + x_0^2 + y_0^2), \\ y = x_0 \sin a(t - t_0) + y_0 \cos a(t - t_0) . \end{cases}$$

Hence the origin is completely stable in this case in the full sense (A), (B).

All of the preceding examples seem to indicate that trigonometric stability is invariably associated with perturbative stability. The following example is of the simplest type which illustrates that this is not the case.

We shall take $n = 4$ with two pairs of variables x_1, y_1, x_2, y_2, and the equations

$$\begin{cases} dx_1/dt = -p_1y_1 + \alpha x_1(x_2^2 + y_2^2), & dy_1/dt = p_1x_1 + \alpha y_1(x_2^2 + y_2^2), \\ dx_2/dt = -p_2y_2 + \beta x_2(x_1^2 + y_1^2), & dy_2/dt = p_2x_2 + \beta y_2(x_1^2 + y_1^2), \end{cases}$$

with α, β of opposite signs. For convenience in dealing with the equations we shall introduce conjugate variables

$$u_i = x_i + y_i\sqrt{-1}, \quad v_i = x_i - y_i\sqrt{-1} \qquad (i = 1, 2),$$

and write the equations in the form

$$\begin{cases} du_1/dt = (p_1\sqrt{-1} + \alpha u_2v_2)u_1, & dv_1/dt = (-p_1\sqrt{-1} + \alpha u_2v_2)v_1, \\ du_2/dt = (p_2\sqrt{-1} + \beta u_1v_1)u_2, & dv_2/dt = (-p_2\sqrt{-1} + \beta u_1v_1)v_2, \end{cases}$$

where α, β are real constants with $\alpha\beta < 0$. It is to be noted that the equations themselves are of conjugate type: that is, when, for instance u_1 and v_1 are interchanged throughout and $\sqrt{-1}$ changed to $-\sqrt{-1}$ in the first pair of equations, the equations are unaltered as a set.

Now multiply the first and second equations of the set just written by v_1, u_1 respectively and add; operate similarly with the second set. There is obtained the pair of equations

$$dw_1/dt = 2\alpha w_1w_2, \qquad dw_2/dt = 2\beta w_1w_2,$$

where $w = u_1v_1$ and $w_2 = u_2v_2$ are positive or zero for all real solutions x_1, y_1, x_2, y_2 of course.

If w_1, for instance, vanishes for any t, these equations show that it will vanish identically, with w_2 a constant.* This gives $u_1 = v_1 = 0$, and the differential equations in u_2, v_2 enable us to write the corresponding solution

$$u_1 = 0, \quad v_1 = 0, \quad u_2 = u_2^{(0)}e^{\sqrt{-1}\,p_2(t-t_0)}, \quad v_2 = v_2^{(0)}e^{-\sqrt{-1}\,p_2(t-t_0)}.$$

Consequently for this type of solution the squared distance from the origin, measured by $w_1 + w_2$ is constant, and the representation is trigonometric.

But let us consider the general solution at distance ϵ from the origin at $t = t_0$ so that

$$w_1^{(0)} + w_2^{(0)} = \epsilon^2, \quad w_1^{(0)} > 0, \quad w_2^{(0)} > 0,$$

* This follows from the fact that the initial values of w_1, w_2 at time t_0 determine the solution uniquely, in virtue of the elementary existence theorems, for $w_1 = 0$, $w_2 =$ const. is such a solution in the case at hand.

thus laying aside the possibility just considered. Introduce a parameter τ for which $d\tau = 2w_1w_2dt$. The equations for w_1, w_2 show that we have

$$w_1 = a(\tau - \tau_0) + w_1^{(0)}, \qquad w_2 = \beta(\tau - \tau_0) + w_2^{(0)},$$

so that w_1 and w_2 vary monotonically and in opposite senses.

Because of the integral relation

$$\beta w_1 - a w_2 = \beta w_1^{(0)} - a w_2^{(0)} = \text{const.},$$

in which β and a are of opposite sign by hypothesis, the positive quantities w_1, w_2, and consequently τ also, vary in a finite interval as t passes from $-\infty$ to $+\infty$.

Let us observe the relation between t and τ further. Taking $t_0 = \tau_0 = 0$ for simplicity of notation, we find

$$2t = \int_0^\tau \frac{d\tau}{(a\tau + w_1^{(0)})(\beta\tau + w_2^{(0)})} .$$

Here the lower limit of integration lies between the two zeros of the denominator of the integrand. Thus τ ranges between these values τ' and τ'' only.

The integral relation above demonstrates that w_1 and w_2 remain less than a fixed multiple of ϵ for all time. Hence we have stability of every order m, in the sense (A).

On the other hand we have from the original equations

$$\begin{cases} du_1/dt = (p_1\sqrt{-1} + aw_2)u_1, & dv_1/dt = (-p_1\sqrt{-1} + aw_2)v_1, \\ du_2/dt = (p_2\sqrt{-1} + \beta w_1)u_2, & dv_2/dt = (-p_2\sqrt{-1} + \beta w_1)v_2, \end{cases}$$

whence we obtain by integration

$$\begin{cases} u_1 = u_1^{(0)} e^{p_1\sqrt{-1}\, t + a\int w_2 dt}, & v_1 = v_1^{(0)} e^{-p_1\sqrt{-1}\, t + a\int w_2 dt}, \\ u_2 = u_2^{(0)} e^{p_2\sqrt{-1}\, t + \beta\int w_1 dt}, & v_2 = v_2^{(0)} e^{-p_2\sqrt{-1}\, t + \beta\int w_1 dt}. \end{cases}$$

Consequently the coordinates u_1, v_1, u_2, v_2, have amplitudes which vary monotonically, but they are essentially trigonometric in the angle variables.

It is obvious (see § 3) that x_1, y_1, x_2, y_2 and u_1, v_1, u_2, v_2 cannot be representable by trigonometric sums as demanded with an error of the second order in ϵ. Consequently property (B) fails to hold for $m = 1$.

Inasmuch as we are not following out very systematically the conditions

for stability of type (A), it may be remarked at this juncture that in general the property (A) will follow if there is an integral in which the initial terms give a definite quadratic form, just as they did in the above example. One is naturally led to conjecture that, in general, condition (A) can only obtain for some order m if there exists a polynomial P in the variables $x_1, \cdots, x_n$ with initial terms yielding a homogeneous form of degree k of definite sign, such that the series on the right in the equation

$$dP/dt = \sum_{j=1}^{n} \partial P/\partial x_j \; X_j$$

will begin with terms of degree $m+k$ at lowest. At any rate the type of argument employed in the following paragraph shows readily that this condition is sufficient for (A).*

Another example illustrating the possibility that (A) may be true for $m=1$ without (B) holding for $m=1$ is furnished by the equations

$$dx/dt = xy^2, \qquad dy/dt = -x^2y.$$

Here $x^2+y^2 = \text{const.}$ is an integral, so that (A) holds for any order m. The general solution is

$$x = \frac{\pm ce^{c^2(t-k)}}{(1+c^2e^{2c^2(t-k)})^{\frac{1}{2}}}, \qquad y = \frac{\pm c}{(1+c^2e^{2c^2(t-k)})^{\frac{1}{2}}},$$

which is not trigonometric in type. However, here the origin is not an isolated point of equilibrium, since all of the points of $x=0$ or $y=0$ are of this type. On account of its degenerate character the example is not as illuminating as the preceding one.

§ 2. *First Order Stability.*

The condition (A) for stability has reference to solutions of the given differential equations (1) lying for $t=t_0$ at a distance ϵ (less than some definite quantity) from the equilibrium point, which will always be taken to be at the origin. Now suppose that we write $x_i = \epsilon y_i$, thus making a linear

* In this connection the reader may be referred to Picard, *Traité d'Analyse*, Vol. 3, chaps. 8, 9, where the fundamental papers of Poincaré, Liapounoff and Levi-Civita are also cited.

transformation of the dependent variables. The equations (1) then take the form

$$dy_i/dt = 1/\epsilon\ X_i(\epsilon y_1, \cdots, \epsilon y_n) = Y_i(y_1, \cdots, y_n, \epsilon) \quad (i = 1, \cdots, n).$$

Since the X_i's vanish at the origin in $x_1, \cdots, x_n$ space, Y_i will be given as power series in $y_1, \cdots, y_n, \epsilon$ without constant term. Moreover the domain of uniform convergence of these new series will certainly extend beyond a distance unity from the origin in $y_1, \cdots, y_n$ space if ϵ is sufficiently small.

Corresponding to any solution of (1) at distance ϵ from the origin to the $x_1, \cdots, x_n$ space, we have a unique solution y_i of the transformed equations at distance unity from the origin in $y_1, \cdots, y_n$ space. Let us write $y_i = a_i$ at $t = t_0$ and regard these initial values a_i as fixed, while ϵ is a parameter which approaches 0. There will be defined a set of solutions y_i varying analytically with ϵ, according to the fundamental existence theorem for ordinary differential equations containing a parameter. For $\epsilon = 0$ the set of functions will satisfy the corresponding equations of variation

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} \frac{\partial X_i}{\partial x_j} \Big|_0 y_j \tag{2}$$

obtained by setting $\epsilon = 0$ in the differential equations for y_i. These are n ordinary linear differential equations with constant coefficients.

Suppose now that we make the hypothesis that the equilibrium point of (1) under consideration is stable in the sense of (A) for $m = 1$. It follows that $y_i = x_i/\epsilon$ is uniformly bounded for $| t - t_0 |$ of order ϵ^{-1}. Consequently, by allowing ϵ to approach 0, we infer that the limiting functions y_i are bounded for all t. Thus for stability of type (A), $m = 1$, it is proved that the equations of variation (2) have all such solutions y_i bounded for all t. But the most general solution of (2) is derived from solutions of this type by multiplication by a constant factor, since the only restriction made was that the sum of the squares $a_1, \cdots, a_n$ is 1. Hence it is a necessary condition for stability of type (A), $m = 1$, that every solution of the equations of variations (2) be bounded for all t.

The known nature of the solutions of (2) enables us immediately to determine the precise significance of this condition.

Suppose first that the roots a of the characteristic equation

$$\left| \frac{\partial X_i}{\partial x_j} \Big|_0 - a\,\delta_{ij} \right| = 0 \qquad (\delta_{ij} = 0,\ i \neq j;\ \delta_{ij} = 1)$$

are distinct from one another. Then there are n particular solutions of the equations of variation which may be written in the form

$$y_1 = c_{1j}e^{a_j t}, \cdots, y_n = c_{nj}e^{a_j t} \qquad (j = 1, \cdots, n).$$

The most general solution is obtained by linear combination of these particular solutions. Here $a_1, \cdots, a_n$ may be real or complex of course, but corresponding real solutions may be obtained by combination from the pair of solutions corresponding to a complex root and its conjugate. It is apparent that such solutions are bounded only in case the expressions $e^{a_j t}$ are bounded. Hence all of the a_i's must be pure imaginary quantities or zero. In this way we see that in the general case of distinct roots of the characteristic equation, it is necessary that the roots occur in pure imaginary pairs except for possible zero roots.

On the other hand in the more complicated case where the roots of the characteristic equation are not distinct, there may arise degenerate solutions not of the above type, but of the form

$$x_1 = c_{1j}t^{l_j}e^{a_j t}, \cdots, \quad x_n = c_{nj}t^{l_j}e^{a_j t} \qquad (l_j > 0),$$

corresponding to the case in which the elementary divisors corresponding to a multiple root a_j are not simple. Evidently these are not bounded. In this way we are led to the following first conclusion: *

For first order stability of type (A) at an equilibrium point it is necessary that all the roots of the characteristic equation belonging to the point be grouped in pure imaginary pairs except for possible zero roots, and that the elementary divisors of the corresponding a matrix be distinct.

We are led at once to ask whether or not this necessary condition for (A) is sufficient. It is easily proved to be sufficient as follows.

Let us effect the linear transformation of (1) (in general complex) which reduces the linear terms in X_i to the canonical form $a_i x_i$. This may always be done in the case of distinct elementary divisors.

In accordance with the well-known theory of real linear transformations this may be accomplished in such a way that pairs of variables x_i and x_j are conjugate which correspond to conjugate roots a_i and a_j of the characteristic equation. We shall take pains to preserve this conjugacy at every subsequent transformation. It should be observed that for such conjugate vari-

* The main facts here are of course well known.

ables the squared 'distance' from the equilibrium point is properly measured by the sum of the squares of the quantities $|\, x_i \,|$.

The transformed equations take the form in the new variables

$$(3_1) \qquad dx_i/dt = a_i x_i + \mathbf{X}_i \qquad (i = 1, \cdots, n),$$

where the power series $\mathbf{X}_i$ in the variables $x_1, \cdots, x_n$ begin with terms of the second degree, while the a_i's are either pure imaginaries or zero. The series $\mathbf{X}_i$, $\mathbf{X}_j$ corresponding to conjugate variables x_i, x_j will be conjugate of course, while the series $\mathbf{X}_i$ corresponding to zero roots of the characteristic equation will be real. If we make the further change of dependent variables

$$x_i = e^{a_i t} z_i \qquad (i = 1, \cdots, n),$$

we find the equations for z_i to be

$$dz_i/dt = \mathbf{X}_i(e^{a_1 t} z_1, \cdots, e^{a_n t} z_n).$$

Since the exponential factors occurring here have absolute value 1, we have clearly for some constant $\mathbf{X}$

$$|\, \mathbf{X}_i \,| \leqq \mathbf{X}\,(|\, z_1 \,|^2 + \cdots + |\, z_n \,|^2) = \mathbf{X} u^2,$$

so long as $y_1, \cdots, y_n$ or $z_1, \cdots, z_n$ are sufficiently small in absolute value. Hence we have

$$\left|\, u \frac{du}{dt} \,\right| \leqq \sum_{j=1}^{n} \left|\, z_j \frac{d\,|\, z_j \,|}{dt} \,\right| \leqq (|\, z_1 \,| + \cdots + |\, z_n \,|)\, \mathbf{X} u^2 \leqq \sqrt{n}\, \mathbf{X} u^3;$$

the elementary inequality for real quantities $b_1, \cdots, b_n$,

$$(b_1 + \cdots + b_n)^2 \leqq \sum_{i,j=1}^{n} |\, b_i b_j \,| \leqq \sum_{i,j=1}^{n} \tfrac{1}{2}(b_i^2 + b_j^2) = n \sum_{j=1}^{n} b_j^2$$

is employed in deducing this result, as well as the obvious relation

$$\left|\, \frac{d\,|\, z_j \,|}{dt} \,\right| \leqq \left|\, \frac{dz_j}{dt} \,\right| \leqq \mathbf{X} u^2.$$

Since $u \neq 0$ we infer

$$-\sqrt{n}\, \mathbf{X} dt \leqq du/u^2 \leqq +\sqrt{n}\, \mathbf{X} dt,$$

whence by integration

$$-\sqrt{n}\, \mathbf{X}(t - t_0) \leqq 1/u_0 - 1/u \leqq \sqrt{n}\, \mathbf{X}(t - t_0) \qquad (t > t_0)$$

with a like inequality for $t < t_0$.

But at $t = t_0$ we have $u = \epsilon$ of course. It appears then that u will not increase to 2ϵ or diminish to $\frac{1}{2}\epsilon$ within an interval

$$| t - t_0 | \leqq \frac{\epsilon^{-1}}{2\sqrt{n}\,\mathrm{X}} \quad .$$

This is what we desired to prove.

The stated necessary condition is sufficient for first order stability of type (*A*).

A slight extension of the preceding proof leads us to the fact that if this condition (A) is satisfied, so that the general solution of the equations of variation is composed of trigonometric sums, then (B) will hold for $m = 1$, i. e., the coordinates can be suitably approximated to by such sums.

Within the same t interval we have $| \mathrm{X}_i | \leqq 4\mathrm{X}u_0^2$ so that from the differential system in $z_1, \cdots, z_n$, the inequalities

$$| z_i - z_i^{(0)} | \leqq 4\mathrm{X}u_0^2 \, | t - t_0 |$$

result by integration, or, in the variables x_i,

$$| x_i - x_i^{(0)} e^{a_i(t-t_0)} | \leqq 4\mathrm{X}u_0^2 \, | t - t_0 |.$$

In other words the n trigonometric sums which constitute the solution of the equations of variation with the same initial values as x_i at t_0 represent x_i with an error not exceeding $4\mathrm{X}\epsilon^2 \, | t - t_0 |$ during an interval $| t - t_0 | \leqq \epsilon^{-1}/2\sqrt{n}\,\mathrm{X}$. A like property must then hold for the solutions of (1) which are related to the solutions of the prepared equations (3_1) by a linear transformation with constant coefficients. Hence we may state the following result:

The stated necessary condition for first order stability is sufficient. If then we let

$$c_{1j}e^{a_j t}, \cdots, c_{nj}e^{a_j t} \qquad (j = 1, \cdots, n)$$

be n linearly independent solutions of the equations of variation (2) *so that* a_j *are pure imaginaries or zero and the determinant* $| c_{ij} | \neq 0$, *and if we form the unique linear combination of these solutions taking on the same values as the solution of* (1) *at* $t = t_0$, *the real trigonometric sums so obtained are of the form*

$$A_0 + \sum_{j=1}^{s} (A_j \cos p_j t + B_j \sin p_j t)$$

where A_j, B_j *are of the first order in* ϵ, *and represent* x_i *with an error not*

exceeding $K\epsilon^2 \mid t - t_0 \mid$ *during an interval* $\mid t - t_0 \mid \leqq L\epsilon^{-1}$ (K, L, *definite positive constants*). *Here* $p_1, \cdots, p_s$ *are the* s *absolute values of the* $2s$ *pure imaginary quantities* a_i, *so that* $2s \leqq n$.

§ 3. *Supplementary Results for First Order Stability.*

It was stated in § 1 that the condition (B) for stability necessitated the condition (A). We propose at first to establish the fact in the case of first order stability. In order to do so we prove a lemma concerning trigonometric sums of the type referred to in the statement of (B).

Lemma Concerning Trigonometric Sums. If a sequence of trigonometric sums

$$A_0 + \sum_{j=1}^{N} (A_j \cos p_j t + B_j \sin p_j t) \qquad (\mid p_i - p_j \mid \geqq P > 0)$$

where P, N are fixed positive constants, approaches a limiting function $\phi(t)$ uniformly in the fixed interval $\mid t - t_0 \mid \leqq T$ then ϕ is itself a trigonometric sum of the same kind.

Proof. The lemma may be proved by induction. Let us first establish it in the case $N = 1$, and let us take $t_0 = 0$. Here it is given that

$$\phi(t) = \lim_{l=\infty} [A_0^{(l)} + (A_1^{(l)} \cos p_1^{(l)} t + B_1^{(l)} \sin p_1^{(l)} t)]$$

where the limit is approached uniformly by the sequence $\phi_l(t)$.

Now the double integration of the term of trigonometric type here occurring simply divides it by $-p_1^{(l)2}$. Hence the sequence of functions

$$\int_0^t \int_0^t \phi_l(t)\, dt^2 + \phi_l(t)/p_1^{(l)2}$$

consists of polynomials of the second degree in t.

Suppose now that $p_1^{(l)}$ has some limit p_1 ($\mid p_1 \mid \geqq P$ since $p_0^{(l)} = 0$) other than ∞ as l becomes infinite. Since the uniform limit of a polynomial of any degree can only be a polynomial of that degree, it is necessary that

$$\int_0^t \int_0^t \phi(t)\, dt^2 + \phi(t)/p_1^2$$

is a polynomial of the second degree in t. Consequently, by double differentiation, we derive

$$d^2\phi/dt^2 + p_1^2\phi = \text{const.},$$

so that ϕ has the form

$$\phi = A_0 + (A_1 \cos p_1 t + B_1 \sin p_1 t) \qquad (\mid p_1 \mid \geqq P)$$

in agreement with the lemma.

If, however, $p_1^{(l)}$ has only the limit ∞ we conclude immediately by a similar method

$$\phi = A_0,$$

in agreement with the lemma.

We remark also that if additional polynomials of the second degree are added to the trigonometric sums, $N = 1$, a similar proof shows that the limit $\phi(t)$ necessarily consists of such a sum augmented by a polynomial of the second degree.

To prove the lemma for $N = 2$ we need only isolate one of the two trigonometric terms with period $2\pi/p_2^{(l)}$ in the approximating trigonometric sums. Proceeding as above we are led to consider

$$\int_0^t \int_0^t \phi_l(t)\, dt^2 + \phi_l(t)/p_2^{(l)2}.$$

This expression lacks the trigonometric terms of the type isolated so that if $p_2^{(l)}$ has a finite limit p_2,

$$\int_0^t \int_0^t \phi(t)\, dt^2 + \phi(t)/p_2^2$$

can be approached uniformly by trigonometric sums of the above type of order $N = 1$ augmented by polynomials of the second degree, and so is itself similarly constituted by the previous remark. By double differentiation, we conclude that ϕ is a trigonometric sum of the second order and of the stated type in this case. When $p_2^{(l)}$ has only the limit ∞ the method can be carried to a conclusion as before.

Evidently we have here the basis of a method of proof by means of induction for $N = 1, 2, \cdots$. Hence the lemma must hold for all values of N.

We may prove that the condition (B) implies (A) for $m = 1$ by making use of this lemma. By hypothesis x_i/ϵ can be approximated to by a trigonometric sum of the prescribed type in any fixed interval $| t - t_0 | \leqq T$ with an error of the first order at most in ϵ. But by choosing x_i properly, x_i/ϵ can be made to approach any solution y_i of the equations of variation having $a_1^2 + \cdots + a_n^2 = 1$. By the lemma then since the functions y_i can be approached uniformly by such trigonometric sums they must be such sums themselves. This means of course that the general solution of the equation of variation is trigonometric, which was seen to be a necessary and sufficient condition for stability of the first order.

It follows that the condition (B) *alone in the case* $m = 1$ *implies condition* (A), *and thus stability of the first order.*

The second result to which I wish to call attention is implicitly contained in the results for trigonometric sums specified in § 2 and has to do with certain amplitudes. Let us formulate the simple notions involved.

The amplitudes are defined to be the absolute values of the variables x_i under consideration, taken as a solution of the prepared equations (3_1). If there are s pairs of conjugate imaginary roots a_i and k zero roots $(n = 2s + k)$ then there are only the amplitudes

$$C_1, \cdots, C_s, \quad C_{s+1}, \cdots, C_{s+k}$$

since conjugate imaginary x_i's have the same absolute value.

It is a fundamental property of the $s + k$ amplitudes that the sum

$$C_1^2 + \cdots C_{s+k}^2$$

serves to measure essentially the squared distance of a point in the original $x_1, \cdots, x_n$ space from the position of equilibrium. In fact the linear transformation relating (1) and (3_1) evidently takes the above sum into a positive definite quadratic form in the values of $x_1, \cdots, x_n$. Any such quadratic form serves to measure the squared distance from the origin as well as the sum of the squares used earlier.

The fact which we desire to establish directly may now be formulated in the following way:

In the case of first order stability the $s + k$ amplitude variables $C_1, \cdots, C_{s+k}$ change only slightly in comparison with the initial total amplitude,

$$\epsilon = (C_1^2 + \cdots + C_{s+k}^2)^{\frac{1}{2}} \qquad (t = t_0)$$

during a time interval $| t - t_0 | \leqq L^* \epsilon^{-1}$ where L^* is to be suitably chosen.

For conjugate x_i and x_j the prepared form (3_1) gives

$$d(x_i x_j)/dt = x_i \mathrm{X}_j + x_j \mathrm{X}_i,$$

while for real x_i we have

$$dx_i^2/dt = 2x_i \mathrm{X}_i.$$

Hence if we write u^2 for the squared amplitude and recall that X_i commence with terms of at least the second degree, we find, as in § 2,

$$du/dt \leqq \mathrm{X} u^2,$$

where X is an appropriately chosen positive constant. Now in § 2 it was shown how from such an inequality we may infer that u will vary between

$\frac{1}{2}u_0$ and $2u_0$ in a suitably taken interval $|t - t_0| \leqq L\epsilon^{-1}$. The same argument permits one to infer that u will vary between $(1-\eta)u_0$ and $(1+\eta)u_0$, η arbitrarily small, if L is chosen small enough. In other words the sum of the squares of the individual amplitudes only varies slightly as compared with the total initial amplitude under these conditions.

But the variables $x_1, \cdots, x_n$ of the prepared equations (3_1) are not uniquely determined, for we may replace $x_1, \cdots, x_n$ respectively by $c_1x_1, \cdots, c_nx_n$ respectively, where $c_1, \cdots, c_n$ are arbitrary positive constants except that c_i and c_j associated with a conjugate pair x_i and x_j are to be equal. In so doing the definition of the amplitudes is changed in that these are replaced by arbitrary constant multiples of themselves. Thus we conclude that, for any arbitrary set of $s+k$ positive constants $\lambda_1, \cdots, \lambda_{s+k}$, the sum

$$\lambda_1 C_1^2 + \cdots + \lambda_{k+s} C_{k+s}^2$$

changes only varies slightly as compared with its initial value for an interval of time of order ϵ^{-1}.

Now choose $s+k$ linearly independent sets $\lambda_1, \cdots, \lambda_{s+k}$ and an arbitrary small value of $\eta > 0$. It will then be possible to choose for each corresponding sum and therefore for all, a constant L^* so small that the ratio of the corresponding sum of squares to the initial sum u_0^2 at $t = t_0$ will vary as little as may be desired for $|t - t_0| \leqq L^*\epsilon^{-1}$.

The theory of linear algebraic equations shows then at once that all of the amplitudes can vary only slightly as compared with the initial total amplitude, as first measured. This yields the result stated.

It will be proved in the following section that when the amplitudes are more carefully defined in the case of first order stability, n even, the variation in these amplitudes remains slight for an interval of time, $|t - t_0| \leqq L^*\epsilon^{-2}$.

§ 4. *Second Order Stability.*

In dealing with second and higher orders of stability, it is convenient to treat separately the cases when the order of the differential system (1) is even $(n = 2s)$ and odd $(n = 2s+1)$. Furthermore we have now to take account of commensurability relations between the characteristic frequencies $p_i/2\pi$.

For the case of even order we confine attention to the case where the inequality

$$i_1 p_1 + \cdots i_s p_s \neq 0$$

holds for sets of integers $i_1, \cdots, i_s$ not all zero. This inequality excludes in particular any zero or equal roots of the characteristic equation. For the purpose of the discussion of second order stability here made, only these special restrictions are necessary.

In the case of odd order there is always one zero root which we take to belong to the coordinate x_n; we designate the $2s$ other pure imaginary or zero roots as in the even order case, and again make the assumption stated above, although only the case of further zero roots or of equal roots needs to be excluded as far as the argument of this section is concerned.

It is with the intention of avoiding undue complexity that restriction to the 'general case' is thus made.

We take up first the case $n = 2s$, and suppose that pairs of conjugate variables are called

$$x_1 = \xi_1, \quad x_2 = \eta_1, \quad \cdots, \quad x_{n-1} = \xi_s, \quad x_n = \eta_s.$$

The prepared form of equations (3_1) may then be written

$$(3) \qquad d\xi_i/dt = p_i \sqrt{-1}\, \xi_i + \Phi_i, \quad d\eta_i/dt = -p_i \sqrt{-1}\, \eta_i + \Psi_i \qquad (i = 1, \cdots, s)$$

where Φ_i, Ψ_i contain no terms of lower than the second degree.

Let us attempt to proceed with the preparation of these second degree terms by means of a further transformation of the form

$$\xi_i = \bar{\xi}_i + \bar{F}_i, \quad \eta_i = \bar{\eta}_i + \bar{G}_i \qquad (i = 1, \cdots, s),$$

where both of the functions $\bar{F}_i$, $\bar{G}_i$ are conjugate homogeneous polynomials of the second degree in $\bar{\xi}_1, \cdots, \bar{\eta}_s$, so that if $\bar{\xi}_i$, $\bar{\eta}_i$ are conjugate imaginaries, ξ_i, η_i $(i = 1, \cdots, s)$ will be, and conversely. The inverse transformation has the form

$$\bar{\xi}_i = \xi_i - F_i + \cdots, \quad \bar{\eta}_i = \eta_i - G_i + \cdots \qquad (i = 1, \cdots, s),$$

where F_i, G_i are the same polynomials in ξ_i, η_i as $\bar{F}_i$, $\bar{G}_i$ in $\bar{\xi}_i$, $\bar{\eta}_i$, and where higher order terms are only indicated. Such a transformation has already been observed not to affect the order of stability (§ 1).

On making this change of variables in the prepared equations (3), we obtain the specific modified equations

$$
\begin{aligned}
\frac{d\xi_i}{dt} &= p_i\sqrt{-1}\,\xi_i + \Phi_i^{(2)} + p_i\sqrt{-1}\,F_i \\
&\quad -\sum_{j=1}^{s} p_j\sqrt{-1}\left(\xi_j\frac{\partial F_i}{\partial \xi_j} - \eta_j\frac{\partial F_i}{\partial \eta_j}\right) + \cdots \\
\frac{d\eta_i}{dt} &= -p_i\sqrt{-1}\,\eta_i + \Psi_i^{(2)} - p_i\sqrt{-1}\,G_i \\
&\quad -\sum_{j=1}^{s} p_j\sqrt{-1}\left(\xi_j\frac{\partial G_i}{\partial \xi_j} - \eta_j\frac{\partial G_i}{\partial \eta_j}\right) + \cdots
\end{aligned}
\tag{4}
$$

where dashes over the ξ_i's and η_i's have been dropped, and where $\Phi_i^{(2)}$, $\Psi_i^{(2)}$ are the second degree components of Φ_i, Ψ_i respectively. The displayed parts contain the complete second degree term in the new equations. Hence this change of variables does not affect the character of the prepared form.

If we let c_i stand for the complete coefficient of a definite term of the second degree

$$c_i\xi_i^{l_1}\eta_1^{m_1}\cdots\xi_s^{l_s}\eta_s^{m_s} \qquad (l_1+m_1+\cdots+m_s=2)$$

in Φ_i, while q_i is the analogous coefficient in F_i, then the corresponding coefficient in the second degree term of the first equation above is

$$c_i - [p_1(l_1-m_1)+\cdots+(l_i-m_i-1)p_i+\cdots + (l_s-m_s)p_s]\sqrt{-1}\,q_i.$$

It is at this point that the hypothesis concerning the general case enters. The coefficient of q_i cannot vanish unless

$$l_1=m_1,\ \cdots,\ l_i=m_i+1,\ \cdots,\ l_s=m_s,$$

Now the argument just stated has not made use of the fact that we are dealing with the simple case $l_1+m_1+\cdots+m_s=2$ when there is evidently no possibility for the coefficient of q_i to vanish. Hence we conclude that without any assumption of second order stability, we may further prepare the equations (3) so that no terms $\Phi_i^{(2)}$, $\Psi_i^{(2)}$ remain.

We may now prove the following result:

In the general case of first order stability, $n=2s$, there will also be second order stability. Let the combined linear and quadratic transformation from ξ_i, η_i to $\bar\xi_i$, $\bar\eta_i$ be made which takes (1) *into the prepared form* (3) *lacking second degree terms in Φ_i, Ψ_i. If we substitute*

$$\bar\xi_i^{(0)}e^{p_i\sqrt{-1}(t-t_0)},\quad \bar\eta_i^{(0)}e^{-p_i\sqrt{-1}(t-t_0)} \qquad (i=1,\cdots,s)$$

for $\bar\xi_i$, $\bar\eta_i$ in the formulas, the trigonometric sums obtained will represent

$x_1, \cdots, x_n$ with an error not exceeding $K\epsilon^3 \mid t - t_0 \mid$ during an interval of time $\mid t - t_0 \mid \leqq L\epsilon^{-2}$.

The details of the proof are very much like those for the analogous result concerning first order stability (§ 2). To begin with, we note that in the prepared form the non-linear terms on the right are of at least the third degree.

Now, using exactly the argument of § 2 except that we can assume $\mid \mathrm{X}_i \mid \leqq \mathrm{X}u^3$, we are able to establish the relation stated.

Suppose now that we define the amplitudes $C_1, \cdots, C_{k+s}$ more precisely by the aid of this especially prepared equation (3). Following a natural modification of the argument made in the preceding section we are at once led to infer the following additional result:

In the case of first, and thus second, order stability ($n = 2s$) the s individual amplitudes $C_1, \cdots, C_s$ may be defined by means of prepared equations (3) in which Φ_i, Ψ_i contain no second degree terms. With this definition, these amplitudes change only slightly in comparison with the total amplitude during a time interval $\mid t - t_0 \mid \leqq L^*\epsilon^{-2}$.

We turn next to the general case of odd order $n = 2s + 1$ for the system of equations (1), and propose to briefly indicate some analogous results.

In this case, besides the $2s$ equations like (3), we have a last equation

$$dx_n/dt = \mathrm{X}$$

where X begins with second degree terms or higher. We attempt to prepare further these equations by a transformation

$$\xi_i = \bar{\xi}_i + \bar{F}_i, \quad \eta_i = \bar{\eta}_i + \bar{G}_i, \quad x_n = \bar{x}_n + \bar{H} \qquad (i = 1, \cdots, s),$$

where $\bar{F}_i, \bar{G}_i, \bar{H}$ are homogeneous polynomials of the second degree. We find precisely as before that we can make $\Phi_i^{(2)} = \Psi_i^{(2)} = 0$ by a proper choice of F_i, G_i, while the second degree terms in X have the form

$$\mathrm{X}^{(2)} - \sum_{j=1}^{s} p_j \sqrt{-1} \left(\xi_j \frac{\partial H}{\partial \xi_j} - \eta_j \frac{\partial H}{\partial \eta_j} \right)$$

If we take typical corresponding terms of the second degree in $\mathrm{X}^{(2)}$ and H, both with exponents $l_1, \cdots, m_s, r$ in $\xi_1, \cdots, \eta_s, x_n$ respectively, and with coefficients c and q, the new analogous coefficient is

$$c - [p_1(l_1 - m_1) + \cdots + p_s(l_s - m_s)] \sqrt{-1}\, q.$$

In the case at hand the only possibility that the coefficient of q vanishes is that some $l_i = m_i = 1$ while the other l_j, m_j must then vanish. Thus the terms $\xi_i\eta_i$ and x_n^2 in $\mathbf{X}^{(2)}$ cannot be removed by this method.

In consequence of this failure in removing certain second degree terms it seems probable in advance that for the case $n = 2s + 1$, first order stability does not imply that of the second order.

A simple example can be devised to show the truth of such a surmise. Take $n = 3$ and write the equations

$$d\xi_1/dt = (p_1\sqrt{-1} + cx_3^2)\xi_1, \quad d\eta_1/dt = (-p_1\sqrt{-1} + cx_3^2)\eta_1,$$
$$dx_3/dt = e\xi_1\eta_1 \qquad (ce > 0),$$

containing one such non-removable term in the last equation.

On writing $u = \xi_1\eta_1$, $v = x_3$ so that $u_0 + v_0^2 = \epsilon^2$, we obtain immediately

$$du/dt = 2cv^2u, \qquad dv/dt = eu,$$

whence

$$d^2v/dt^2 = 2cv^2\, dv/dt.$$

This is immediately integrable, and yields

$$t - t_0 = 3/2c \int_{v_0}^{v} \frac{dv}{v^3 - v_0^3 + (3e/2c)u_0}.$$

Suppose now that we take $v_0 = 0$ and substitute

$$v = \left(\frac{3eu_0}{2c}\right)^{1/3} w.$$

We find

$$t - t_0 = (3/2c)^{1/3}u_0^{-2/3}e^{-2/3} \int_0^w \frac{dw}{w^3 + 1}.$$

Taking the upper limit $w = +\infty$ we conclude that the corresponding interval of time is of order $\epsilon^{-4/3}$, since the integral converges. Thus v does not remain finite in an interval of order $\epsilon^{-4/3}$, and property (A) does not hold for $m = 2$. Hence in this special example the presence of non-removable terms is associated with instability of the second order.

We proceed now to prove that (B) can only hold if there are no such non-removable terms in the completely prepared form. Here we use once more the lemma on trigonometric sums of the preceding section.

Let

$$\xi_i = \epsilon\xi_i^* + \epsilon^2\xi_i^{**} + \cdots, \qquad \eta_i = \epsilon\eta_i^* + \epsilon^2\eta_i^{**} + \cdots,$$
$$x_n = \epsilon x_n^* + \epsilon^2 x_n^{**} + \cdots$$

be that solution of the prepared equations for which

$$\xi_i = a_i^{(0)}\epsilon, \quad \eta_i = b_i^{(0)}\epsilon, \quad x_n = c^{(0)}\epsilon \qquad (t = t_0).$$

Here $a_i^{(0)}$ and $b_i^{(0)}$ are conjugate imaginary quantities, and $c^{(0)}$ real, with

$$a_1^{(0)}b_1^{(0)} + \cdots + a_s^{(0)}b_s^{(0)} + c^{(0)2} = 1.$$

Also ξ_i^*, η_i^*, x_n^* are solutions of the equations of variation

$$\xi_i^* = a_i^{(0)}e^{p_i\sqrt{-1}(t-t_0)}, \quad \eta_i^* = b_i^{(0)}e^{-p_i\sqrt{-1}(t-t_0)}, \quad x_n^* = c^{(0)}.$$

We consider now the second terms in these series.

By differentiating the prepared equations twice as to ϵ and setting $\epsilon = 0$, we find

$$\frac{d\xi_i^{**}}{dt} = p_i\sqrt{-1}\,\xi_i^{**}, \quad \frac{d\eta_i^{**}}{dt} = -p_i\sqrt{-1}\,\eta_i^{**},$$
$$\frac{dx_n^{**}}{dt} = c_1\xi_1^*\eta_1^* + \cdots + c_s\xi_s^*\eta_s^* + bx_n^{*2},$$

where the terms on the right in the last equation arise from the non-removable terms of **X**. Furthermore for $t = t_0$, $\xi_i^{**} = \eta_i^{**} = x_n^{**} = 0$, since, for example, ξ_i^{**} is the second partial derivative of ξ_i as to ϵ, and ξ_i is constantly equal to $a_i^{(0)}\epsilon$ at $t = t_0$.

Thus we infer that ξ_i^{**}, η_i^{**} are identically zero, while from the last equation

$$x_n^{**} = [c_1a_1^{(0)}b_1^{(0)} + \cdots + c_sa_s^{(0)}b_s^{(0)} + bc^{(0)2}]\ (t - t_0).$$

But x_n can be approximated by a trigonometric sum of the stated type to the third order in ϵ, by hypothesis. Thus $(x_n - c^{(0)}\epsilon)/\epsilon^2$ can be similarly approximated to the first order in ϵ. Consequently the limit of this quotient, x_n^{**}, is such a sum by the lemma. This can only be a trigonometric sum for all choices of $a_i^{(0)}$, $b_i^{(0)}$, $c^{(0)}$ if $c_i = d = 0$ $(i = 1, \cdots, s)$, which we wished to prove.

In the general case $n = 2s + 1$ a necessary and sufficient condition for second order stability is that a prepared form (3) *exists with a supplementary equation*

$$dx_n/dt = \mathbf{X},$$

where Φ_i, Ψ_i, $\mathbf{X}$ begin with terms of at least the third degree. Thence follow results concerning the representation by trigonometric sums, and concerning the variation of the $s + 1$ suitably defined amplitudes, which are entirely analogous to the results in the case $n = 2s$.

In fact, we have reached a prepared form in the case $n = 2s + 1$ with the same characteristics as that obtained in the case $n = 2s$ earlier, and can apply the same arguments with obvious modifications.

§ 5. *Third and Higher Order Stability.*

We propose now to assume stability of the third order, taking $n = 2s$ first, and then, by making use of condition (B) only, to obtain a necessary prepared form of equations, which can readily be proved sufficient for stability of the third order. An entirely analogous treatment can be made for $n = 2s + 1$.

By passing thus from the case of second to third order stability as well as from the first to the second, the general results can be formulated.

Starting with the prepared form (3) we now attempt to remove as many of the third order terms as possible by writing

$$\xi_i = \bar{\xi}_i + \bar{F}_i, \quad \eta_i = \bar{\eta}_i + \bar{G}_i \qquad (i = 1, \cdots, s)$$

where now $\bar{F}_i$, $\bar{G}_i$ are homogeneous polynomials of the *third* degree in $\bar{\xi}_1$, $\cdots$, $\bar{\eta}_s$. By exactly the same formal work as in our treatment of second order stability, it turns out that all such terms may be removed except for terms of the form $U_i\xi_i$, $V_i\eta_i$ in the ith pair of equations, where U_i, V_i are linear combinations of the s products $\xi_1\eta_1, \cdots, \xi_s\eta_s$. Thus we have

$$\begin{aligned} d\xi_i/dt &= (p_i\sqrt{-1} + U_i)\xi_i + \cdots, \\ d\eta_i/dt &= (-p_i\sqrt{-1} + V_i)\eta_i + \cdots \qquad (i = 1, \cdots, s), \end{aligned}$$

where the terms not explicitly written are at least of the fourth degree. From these equations we derive

$$\frac{d(\xi_i\eta_i)}{dt} = (U_i + V_i)\xi_i\eta_i + \cdots \qquad (i = 1, \cdots, s),$$

where the omitted terms are at least of the fifth degree.

Now we expand ξ_i, η_i in power series in ϵ as before. To determine the coefficients in this series we proceed precisely as in the case of second order stability and find series in ϵ which we write for $i = 1, \cdots, s$,

$$\begin{aligned} \xi_i &= \epsilon a_i^{(0)} e^{p_i\sqrt{-1}(t-t_0)} + * + \epsilon^3\xi_i^{**} + \cdots, \\ \eta_i &= \epsilon b_i^{(0)} e^{-p_i\sqrt{-1}(t-t_0)} + * + \epsilon^3\eta_i^{**} + \cdots. \end{aligned}$$

In virtue of the hypothesis of third order stability and the use of our lemma on trigonometric sums, we conclude that ϕ_i^*, where

$$\phi_i = \xi_i\eta_i = \epsilon^2 a_i^{(0)} b_i^{(0)} + \epsilon^4 \phi_i^* + \cdots,$$

are trigonometric sums.

If now we substitute these different series in the equations above in ϕ_i, and compare terms in ϵ^4 we find simply

$$d\phi_i^*/dt = [U_i(a_1^{(0)}b_1^{(0)}, \cdots,) + V_i(a_1^{(0)}b_1^{(0)}, \cdots)]\, a_i^{(0)} b_i^{(0)}.$$

Consequently unless $U_i + V_i$ vanishes identically for $i = 1, \cdots, s$, the functions ϕ_i^* will not be trigonometric sums but will be linear functions of t, which is not possible.

This gives us the necessary form of prepared equations in the case of third order stability, $n = 2s$,

$$(5) \qquad \begin{cases} d\xi_i/dt = M_i\xi_i + \Phi_i, \\ d\eta_i/dt = -M_i\eta_i + \Psi_i, \end{cases} \qquad (i = 1, \cdots, s),$$

where M_i has a constant term $p_i\sqrt{-1}$ and linear terms in the s products $\xi_1\eta_1, \cdots, \xi_s\eta_s$, and where Φ_i, Ψ_i begin with of at least fourth degree.

Our first general conclusion, based on a step-by-step process to higher and higher orders, is thus the following.

For $n = 2s$ a necessary condition that (B) *may hold for any odd order m of stability is that the equations may be transformed to the form* (5) *in which the functions M_i are polynomials in the s products $\xi_1\eta_1, \cdots, \xi_s\eta_s$ of degree $\frac{1}{2}(m-1)$ at most, with initial terms $\sqrt{-1}\,p_i$, and having pure imaginary coefficients throughout,* and where Φ_i, Ψ_i commence with terms of degree at least $m+1$. In this case Φ_i, Ψ_i may be made to commence with terms of degree $m+2$ at lowest without further assumption.*

To show that this form (5) induces $(m+1)$st order stability (m odd) we obtain first of course

$$\left| \frac{d(\xi_i\eta_i)}{dt} \right| \leqq \mathrm{X}u^{m+3},$$

where $u^2 = \xi_1\eta_1 + \cdots + \xi_s\eta_s$ and thence

$$\left| \frac{du}{dt} \right| \leqq \tfrac{1}{2}s\mathrm{X}u^{m+2}.$$

* These are pure imaginary quantities since the variables ξ_i, η_i are self-conjugate, and $M_i\xi_i$ must be conjugate to $-M_i\eta_i$.

Here X is a fixed constant. From this inequality we infer that u remains of the first order in ϵ for the interval $|t-t_0| \leqq L\epsilon^{-(m+1)}$ by the same method already employed for $m=1$.

This gives us the property (A) as a consequence of the prepared form.

Now we deduce also from the same inequality

$$|\xi_i\eta_i - \xi_i^{(0)}\eta_i^{(0)}| \leqq k\mathrm{X}\epsilon^{m+3}|t-t_0|$$

in this same interval. On going back to the prepared form this yields at once

$$|d\xi_i/dt - M_i^{(0)}\xi_i| \leqq K\epsilon^{m+3}|t-t_0| + N\epsilon^{m+2}$$

in which $M_i^{(0)}$ designates what M_i becomes when ξ_i, η_i are replaced by $\xi_i^{(0)}$, $\eta_i^{(0)}$. Hence we infer

$$|d(\xi_i e^{-M_i^{(0)}(t-t_0)})/dt| \leqq K\epsilon^{m+3}|t-t_0| + N\epsilon^{m+2},$$

and finally

$$|\xi_i - \xi_i^{(0)}e^{M_i^{(0)}(t-t_0)}| \leqq K\epsilon^{m+3}\frac{(t-t_0)^2}{2} + N\epsilon^{m+2}|t-t_0|$$

in an interval of the order of $\epsilon^{-(m+1)}$.

Hence we infer in passing that property (B) holds as well, and obtain additional properties of the approximate trigonometric representation.

The necessary condition of the prepared form (5) *insures stability in the complete sense that* (A) *and* (B) *hold, and so is sufficient. The explicit solution of the prepared form with terms* Φ_i, Ψ_i *omitted, and with initial values* $\xi_i^{(0)}$, $\eta_i^{(0)}$ *chosen to correspond to the initial values* $x_1^{(0)}, \cdots, x_n^{(0)}$ *of* $x_1, \cdots, x_n$ *yields trigonometric sums by means of the composite transformation which represent* x_i *with an error not exceeding*

$$K\epsilon^{m+3}\frac{(t-t_0)^2}{2} + N\epsilon^{m+2}|t-t_0|$$

during a time interval

$$|t-t_0| \leqq L\epsilon^{-(m+1)}$$

where ϵ *denotes the initial distance from the origin and where K, L, N are positive constants.*

If we define the amplitudes $C_1, \cdots, C_s$ of a solution by means of the prepared form it is clear that we can establish the following result:

The amplitudes $C_1, \cdots, C_s$ *vary slightly in comparison with the total*

initial amplitude ϵ *during an interval* $|t - t_0| \leqq L^*\epsilon^{-(m+1)}$, L^* *being a suitably chosen positive constant.*

There remains now to refer briefly to the case n odd. It is only the handling of the last equation in x_n that needs to be referred to. In passing from the second order case to the third it is seen at once that the third order terms in this equation may be entirely removed without any assumption concerning third order stability. The third order terms in the other equations can be treated as in the case n even.

In the case of mth order stability, $n = 2s + 1$, *the prepared form is*

$$(6) \qquad d\xi_i/dt = M_i\xi_i + \Phi_i, \qquad d\eta_i/dt = -M_i\eta_i + \Psi_i, \qquad (i = 1, \cdots, s)$$
$$dx_n/dt = \mathbf{X},$$

where M_i *are functions of the products* $\xi_i\eta_i$ *and of* x_n *of degree* m *at most in* $\xi_1, \cdots, \eta_s$, x_n, and Φ_i, Ψ_i, $\mathbf{X}$ *have no terms of degree lower than* $m + 1$. *From this form, the properties* (A) *and* (B), *suitable trigonometric sums, and the slow variation of the amplitudes follow just as in the case* $n = 2s$.

§ 6. *Formal Theory of Complete Stability.*

In this section we shall consider equations (1) of even order $n = 2s$, and for convenience we shall write the equations in the form

$$(7) \qquad d\xi_i/dt = \Phi_i(\xi_1, \cdots, \eta_s), \quad d\eta_i/dt = \Psi_i(\xi_1, \cdots, \eta_s) \qquad (i = 1, \cdots, s)$$

where the conjugate power series Φ_i, Ψ_i are without constant terms and where the first degree terms in Φ_i and Ψ_i are $p_i\sqrt{-1}$ and $-p_i\sqrt{-1}$ respectively. We suppose furthermore that we are confronted by the general case as defined in § 3.

In other words, of all possible equations (1) we are dealing with those typical of first order stability.

It is obvious that any formal power series transformation

$$(8) \qquad \xi_i = \bar{\xi}_i + \bar{F}_i, \qquad \eta_i = \bar{\eta}_i + \bar{G}_i \qquad (i = 1, \cdots, s),$$

with $\bar{F}_i$ and $\bar{G}_i$ conjugate series in $\bar{\xi}_i$, $\bar{\eta}_i$, carries the system into another formal system of the same type, in which $\bar{\Phi}_i$, $\bar{\Psi}_i$ may not be convergent series, if $\bar{F}_i$, $\bar{G}_i$ are not. It is apparent that we have here a formal group leaving

(7) invariant, which will be perfectly consistent with itself in the sense that the result of carrying out two successive transformations is the same as that derived by the composite transformation.*

Now as already proved we can remove the second degree terms in Φ_i, Ψ_i by using $\bar{F}_i$, $\bar{G}_i$ which are homogeneous quadratic polynomials. Following this with a transformation in which $\bar{F}_i$, $\bar{G}_i$ are cubic, we can remove all terms of the third degree save those in Φ_i and Ψ_i of the respective types $U_i\xi_i$ and $V_i\xi_i$ where U_i, V_i are linear in the s variables $\xi_1\eta_1, \cdots, \xi_s\eta_s$. All of this goes just as in our discussion of stability in the earlier sections, and we arrive at the following first conclusion by an infinite succession of steps.

Under the formal group the equations (7) *may be given the form*

$$d\xi_i/dt = U_i\xi_i, \quad d\eta_i/dt = V_i\eta_i \qquad (i = 1, \cdots, s), \tag{9}$$

where U_i, V_i are power series in the s products $\xi_1\eta_1, \cdots, \xi_s\eta_s$.

This result is extremely interesting from the formal point of view for the following reason:

If we write $u_i = \xi_i\eta_i$ then we have the formal equations in $u_1, \cdots, u_s$

$$du_i/dt = [U_i(u_1, \cdots) + V_i(u_1, \cdots)]u_i \qquad (i = 1, \cdots, s), \tag{10}$$

thus yielding an associated formal differential system of one half the order of the original system (7).

We propose next to ask what is the most general transformation of the group (8) carrying the normal form (9) into the normal form. Let us consider terms of the second order in $\bar{F}_i$, $\bar{G}_i$ which alone can modify those of the second order in U_i, V_i. According to the discussion carried out in the treatment of second order stability, these are uniquely determined by the initial and final forms of (9). But in neither are there terms of the second degree. Hence $\bar{F}_i$, $\bar{G}_i$ lack second degree terms.

Pass next to third degree terms in U_i, V_i which are only affected by the terms of equal or less degree in $\bar{F}_i$, $\bar{G}_i$. Here we find as in dealing with third order stability that since $U_i\xi_i$ and $V_i\eta_i$ have only terms of the third degree which involve a factor ξ_i and η_i together with a product $\xi_i\eta_i$ linearly, $\bar{F}_i$, $\bar{G}_i$ contain only terms of the third degree of the same character. Thus we may continue indefinitely, and we infer

* It is to be observed this group admits of a slight extension in that the initial terms $\bar{\xi}_i, \bar{\eta}_i$ in the series for ξ_i, η_i might be replaced by $c_i\xi_i$, $d_i\eta_i$ respectively (c_id_i being self-conjugate of course).

The most general transformation (8) *which carries the normal form into itself has the form*

$$\xi_i = \bar{\xi}_i \bar{F}_i^*, \qquad \eta_i = \bar{\eta}_i \bar{G}_i^* \qquad (i = 1, \cdots, s) \tag{11}$$

where F_i^*, G_i^* *are series in the products* $\xi_1\eta_1, \cdots, \xi_s\eta_s$ *with constant terms* **1**. *The form of this transformation indicates that the family of formal 'surfaces'* $Q(u_1, \cdots, u_s) = const.$ *is invariant under the group.*

The general invariant theory of equations (7) is less simple than that of the special class of equations possessing complete formal stability.

A necessary and sufficient condition for complete stability of a system of equations (7) *is that* $U_i + V_i = 0$ $(i = 1, \cdots, s)$ *so that the equations take a normal form*

$$d\xi_i/dt = M_i\xi_i, \qquad d\eta_i/dt = -M_i\eta_i \qquad (i = 1, \cdots, s) \tag{12}$$

where $M_1, \cdots, M_s$ *are pure imaginary formal power series in the* s *products* $\xi_1\eta_1, \cdots, \xi_s\eta_s$ *with initial terms* $p_1\sqrt{-1}, \cdots, p_s\sqrt{-1}$ *respectively.*

For it is precisely a result of the earlier sections that stability of the mth order yields this normal form to the terms of order m. Also the new transformations employed at successive stages ($m = 1, 2, 3, \cdots$) do not affect terms of lesser degree. Hence we obtain a limiting formal transformation which takes the equations into the given form. Conversely, if there is such a form, the actual transformation obtained from the formal transformation by breaking off at the term of $(m+1)$st degree will yield the same form of equations out to terms of the same degree so that the prepared form is actually obtained, and thus stability of any arbitrary order m is ensured.

These series M_i must be pure imaginary since the products $\xi_i\eta_i$ involved are self-conjugate.

For this system (12) we have the integrals $\xi_i\eta_i = \text{const.}$ Hence in the most general transformation (11) which keeps the normal form (12) unaltered we have $\bar{\xi}_i\bar{\eta}_i = \text{const.}$ also. On this account it is possible to determine the transformed equations at once. We find

$$\frac{d\bar{\xi}_i}{dt} = \frac{1}{\bar{F}_i^*}\,\frac{d\xi_i}{dt} = \frac{M_i\xi_i}{\bar{F}_i^*} = M_i\bar{\xi}_i$$

with like equations in $\bar{\eta}_i$.

The most general transformation (11) *leaves the normal equations* (12)

unaffected except that in each M_i the products $\xi_i\eta_i$ are replaced by $\xi_i\bar{\eta}_i\bar{F}_i^*G_i^*$. Thus the subgroup for which $G_i^* = 1/F_i^*$ leaves the equations unaffected in every respect.

The question naturally presents itself as to what are the invariants of the equations of completely stable type. The reply is obviously the following:

Reduce the given equations with the property of complete stability to the normal form (12), *thus obtaining s associated series*

$$M_1(u_1, \cdots, u_s), \cdots, M_s(u_1, \cdots, u_s).$$

The invariants of the completely stable type of equations will be the invariants of these forms under arbitrary transformations

$$u_i = \bar{u}_i\bar{f}_i \qquad (i = 1, \cdots, s)$$

where the $\bar{f}_i$ are arbitrary real power series except that they must have a constant term 1.

In this normal form the equations can be at once integrated precisely because $\xi_i\eta_i = \xi_i^{(0)}\eta_i^{(0)}$ give s integrals for $i = 1, \cdots, s$.

The integrated form of (12) *is*

$$(13) \qquad \xi_i = \xi_i^{(0)}e^{M_i^{(0)}(t-t_0)}, \quad \eta_i = \eta_i^{(0)}e^{-M_i^{(0)}(t-t_0)}, \quad (i = 1, \cdots, s).$$

When these expressions are substituted in the formal series giving $x_1, \cdots, x_n$ in terms of $\xi_1, \cdots, \eta_s$ the general formal trigonometric solution is obtained, involving $\xi_1^{(0)}, \cdots, \eta_s^{(0)}$ as the n arbitrary constants of integration.

The products $\xi_i\eta_i$ give real power series in the s variables $x_1, \cdots, x_n$ evidently beginning with a quadratic form which is the product of two conjugate linear factors. This result may be stated as follows:

If the equations (1) *have the property of complete stability there exist s first integrals $\phi_i = c_i$ $(i = 1, \cdots, s)$, given by formal series beginning with a second degree term which is the product of two conjugate linear factors, forming in all $n = 2s$ linearly independent linear expressions.*

Of course an arbitrary $Q(u_1, \cdots, u_s) = \text{const.}$ also yields a first integral, for any real power series Q.

It remains to say a word about the case of odd order $n = 2s + 1$. It is the normal form that is particularly to be noted. This may be written

$$(12') \qquad d\xi_i/dt = M_i\xi_i, \qquad d\eta_i/dt = -M_i\eta_i, \qquad dx_n/dt = 0 \qquad (i = 1, \cdots, s).$$

Here M_i is a power series in the s products $\xi_1\eta_1, \cdots, \xi_s\eta_s$ and in x_n only. Thus the equations separate into a system of the type obtained in the case of even order together with an equation yielding $x_n = \text{const.}$ for the remaining variable.

The normal form of equation for the completely stable case, $n = 2s + 1$, consists of the equations (12′). These equations display the slight difference between the odd and even order case. In the odd order case there exists a real formal series integral $\phi = const.$ (corresponding to $x_n = const.$) containing a first degree term, as well as s integrals $\phi_i = const.$

§ 7. *Variational Principles and the Equations of Dynamics.*

Let $P_1, \cdots, P_n, Q$ be $n + 1$ real analytic functions ($n = 2s$) of $x_1, \cdots, x_n$ in a certain open connected continuum under consideration, and let C^* be an arc of a curve with continuously turning tangent lying within this continuum given by

$$x_i = x_i^*(t) \qquad (t_0 \leqq t \leqq t_1)$$

for $i = 1, \cdots, n$, where x_i^* are continuous functions of t with continuous derivatives. In this case the integral

$$(14) \qquad I = \int_{t_0}^{t_1} \{P_1dx_1 + \cdots + P_ndx_n + Qdt\}$$

has a certain numerical value I^* along the curve C^*. Now suppose that

$$x_i = x_i^*(t, \lambda) \qquad (t_0 \leqq t \leqq t_1)$$

for $i = 1, \cdots, n$ represents any arbitrary one-parameter family of such curves which gives C^* for $\lambda = 0$, and has the same end points as C^* for all values of λ. Suppose furthermore that the functions $x_i^*(t, \lambda)$ are restricted to have continuous partial derivatives as to t and λ. A family

$$x_i = x_i^*(t) + \lambda\,\delta x_i(t) \qquad (i = 1, \cdots, n)$$

will fulfill these conditions if the δx_i are continuous together with their derivatives and vanish at $t = t_0$ and $t = t_1$, while λ is taken sufficiently small. Under these circumstances I becomes a function $I(\lambda)$ when $x_i^*(t, \lambda)$ are substituted for x_i under the sign of integration, and this function $I(\lambda)$ will have a continuous derivative as to λ whose value for $\lambda = 0$ we denote by δI. This

variation δI may vanish identically along a curve C^*, i. e., $\delta I^* = 0$, and it turns out by direct inspection of the expression for δI that the condition is precisely that the coordinates $x_1^*, \cdots, x_n^*$ satisfy the set of n Pfaffin differential equations,

$$(15) \qquad \sum_{j=1}^{n} \left(\frac{\partial P_i}{\partial x_j} - \frac{\partial P_j}{\partial x_i} \right) \frac{dx_j}{dt} - \frac{\partial Q}{\partial x_i} = 0 \qquad (i = 1, \cdots, n).$$

All this amounts merely to the statement of some elementary principles in the calculus of variations as applied to a particular and very important type of integral.

On account of the fact that the vanishing of δI along a curve in this sense is obviously invariant of the particular coordinate system, it follows that if we make a change of variables

$$(16) \qquad x_i = \phi_i(\bar{x}_1, \cdots, \bar{x}_n) \qquad (i = 1, \cdots, n),$$

where the correspondence between $x_1, \cdots, x_n$ and $\bar{x}_1, \cdots, \bar{x}_n$ space is one-to-one and analytic along the curve, the new equations can be obtained by determining $P_1, \cdots, P_n, Q$ by substitution under the integral sign, and then writing down the new corresponding Pfaffian equations. But the formal relation so established can of course be stated and proved directly, without intervention of the principle of variation.

Suppose now that we consider formal (i. e. convergent or divergent) series for P_i, Q in the neighborhood of the origin in $x_1, \cdots, x_n$ space, and also the corresponding formal equations (15), and suppose that we effect the formal transformation (16) in which ϕ_i are given by similar series without constant terms for which the formal Jacobian has a constant term not zero. The new equations can be obtained by the same rule of transformation as before, and in general the result of more than one such transformation is of course the same as that for the composite formal transformation. These statements can be proved by first taking the actual equations obtained by breaking off the series for P_i, Q at terms of high degree m, while actual transformations are obtained by breaking off the series for ϕ_i in the same way. By considering the relations involved as m becomes infinite, the stated formal results follow at once.

I state these facts at some length since it is important to realize that in this way the condition $\delta I = 0$ yields a perfectly definite system of formal differential equations (15) subject to the same formal rules as in the case of actual convergence.

This point of view admits of a slight extension which will come into play in our first theorem. Suppose that P_i, Q are generalized to the respective forms P_i/D, Q/D, where P_i, Q, D are formal power series, but where D need not contain a constant term. However, nevertheless we obtain an extended Pfaffian system

$$(15') \qquad \sum_{j=1}^{n} \left\{ \left[D\frac{\partial P_i}{\partial x_j} - P_i\frac{\partial D}{\partial x_j} \right] - \left[D\frac{\partial P_j}{\partial x_i} - P_j\frac{\partial D}{\partial x_i} \right] \right\} \frac{dx_j}{dt} - D\frac{\partial Q}{\partial x_i} + Q\frac{\partial D}{\partial x_i} = 0.$$

The equations obtained will be independent of any formal factor μ introduced in all of the numerators and denominators of P_i/D, Q/D. Thus the formal significance of the condition $\delta I = 0$ is maintained even in this case.

If the equations (1) *are of completely stable type at the origin, and if* ξ_i, η_i, M_i $(i = 1, \cdots, s)$ *are the series in* $x_1, \cdots, x_n$ *which appear in the normal form* (12), *then the equations* (1) *may be expressed in the extended Pfaffian form*

$$(17) \qquad \delta \int_{t_0}^{t_1} \left\{ \sum_{j=1}^{s} \frac{M_j}{\xi_j\eta_j} (\eta_j d\xi_j - \xi_j d\eta_j) - \sum_{j=1}^{s} M_j{}^2\, dt \right\} = 0.$$

According to what has been said above, it is only necessary to verify that (17) leads to the equations (12) when $\xi_1, \cdots, \eta_s$ are taken as the dependent variables. This may be immediately verified if the conditions $dM_i = 0$ $(i = 1, \cdots, s)$ are equivalent to $d\xi_i\eta_i = 0$, as we shall take them to be. A still more formal attack is to change variables in (12), writing $u_i = M_i$, $v_i = \log \xi_i/\eta_i$ $(i = 1, \cdots, s)$. The condition (17) becomes

$$(17') \qquad \delta \int_{t_0}^{t_1} \left\{ \sum_{j=1}^{s} u_j{}^2\, dt - \sum_{j=1}^{s} u_j{}^2\, dt \right\} = 0,$$

giving the desired equations

$$du_i/dt = 0, \quad dv_i/dt = 2u_i, \qquad (i = 1, \cdots, s)$$

if we grant the legitimacy of the formal processes involved.

The equations given by (17') are of Hamiltonian form with u_i, v_i as conjugate variables and with principal function $H = \sum_{j=1}^{s} u_j{}^2$. Or, better still, we can remain within the non-extended type of Hamiltonian equations in case M_i, aside from its constant term which does not effectively enter in (17),

is divisible by $\xi_i\eta_i$. In this case $(M_j - p_j\sqrt{-1})/\xi_j\eta_j$ is an admissible power series, and (17) may be modified to the equivalent form

$$\delta \int_{t_0}^{t_1} \left\{ \sum_{j=1}^{s} \sqrt{-1}\,(\bar{\eta}_j d\xi_j - \bar{\xi}_j d\bar{\eta}_j) - \sum_{j=1}^{s} M_j{}^2 dt \right\} = 0,$$

giving equations of the Hamiltonian form, if we take

$$\bar{\xi}_i = \xi_i \left(\frac{M_i - p_i\sqrt{-1}}{\xi_i\eta_i\sqrt{-1}}\right)^{\frac{1}{2}}, \qquad \bar{\eta}_i = \eta_i \left(\frac{M_i - p_i\sqrt{-1}}{\xi_i\eta_i\sqrt{-1}}\right)^{\frac{1}{2}} \tag{18}$$

as conjugate variables and $\frac{1}{2}\sum_{j=1}^{s} M_j{}^2/\sqrt{-1}$ as principal function. Also, this choice of variables will lie in the original formal group in case M_i contains a term $c_i\xi_i\eta_i$, $c_i \neq 0$, of the first degree. Of course it can contain no other such terms in this case. These results may be summarized as follows.

The equations (1) *of completely stable type can be made to assume the Hamiltonian form, at least if we admit the legitimacy of certain extended formal transformations. In any case if* $M_i - p_i\sqrt{-1}$ *is divisible by* $\xi_i\eta_i$ *for* $i = 1, \cdots, s$, *with linear term* $c_i\xi_i\eta_i$ $(c_i \neq 0)$ *then the equations* (1) *can be made to take Hamiltonian form by means of* (18), *without any use of such extended transformations.*

The last part of this result is obvious from the normal form (12) which becomes of the very special type

$$d\xi_i/dt = (p_i + \xi_i\eta_i)\sqrt{-1}\,\xi_i, \quad d\eta_i/dt = (-p_i - \xi_i\eta_i)\sqrt{-1}\,\eta_i \qquad (i = 1, \cdots, s)$$

if use be made of the special variables (18).

Consider now a Hamiltonian system (1) of general type with equilibrium point at the origin, which can be written

$$\frac{du_i}{dt} = \frac{\partial H}{\partial v_i}, \quad \frac{dv_i}{dt} = -\frac{\partial H}{\partial u_i} \qquad (i = 1, \cdots, s)$$

in which the lowest degree terms in H are

$$\sqrt{-1}\,(p_1u_1v_1 + \cdots + p_su_sv_s)$$

where the numbers p_i arise in the same way as for any equilibrium point with first order stability. The variables u_i, v_i are conjugate imaginary variables of course so that a preliminary linear transformation of the real variables is presupposed.

It is well-known that the transformation for arbitrary K,

$$u_i = \frac{\partial K}{\partial v_i}, \qquad \overline{v}_i = \frac{\partial K}{\partial \bar{u}_i} \qquad (i = 1, \cdots, s)$$

preserves the Hamiltonian form, i. e. that the transformed equations are obtained by replacing u_i, v_i by their series in $\bar{u}_i$, $\overline{v}_i$ in H to obtain $\bar{H}$. We shall assume that

$$K = \bar{u}_1 v_1 + \cdots + \bar{u}_s v_s + \cdots$$

where the terms not explicitly written are of higher than the second degree. If K has only the first term, the transformation becomes the identity

$$u_i = \bar{u}_i, \quad \overline{v}_i = v_i \qquad (i = 1, \cdots, s).$$

These transformations form a group.

We propose first to take K of the third degree and thus get rid of the third degree terms in H; next to take K as the sum of the first quadratic term and a homogeneous fourth degree term and thus eliminate all the fourth degree terms of H except those quadratic in the s products $u_1 v_1, \cdots, u_s v_s$; and so on. Thus by an indefinite sequence of such transformations we propose to make H a function of the s products $u_1 v_1, \cdots, u_s v_s$.

Now the first of these transformations is

$$u_i = \bar{u}_i + \frac{\partial K^*}{\partial v_i}, \quad \overline{v}_i = v_i + \frac{\partial K^*}{\partial \bar{u}_i} \qquad (i = 1, \cdots, s),$$

in which the variables on the right of each equation are $\bar{u}_i$, v_i, and K^* is a homogeneous cubic in these variables. If we express u_i, v_i in terms of $\bar{u}_i$, $\overline{v}_i$ they take the form

$$u_i = \bar{u}_i + \frac{\partial \bar{K}^*}{\partial \overline{v}_i} + \cdots, \quad v_i = \overline{v}_i - \frac{\partial \bar{K}^*}{\partial \bar{u}_i} + \cdots,$$

where $\bar{K}^*$ is K^* with the variables $\bar{u}_i$, v_i replaced by $\bar{u}_i$, $\overline{v}_i$, and the complete first and second degree terms are explicitly written.

Consequently $\bar{H}^{(3)}$ as obtained by the usual expansion in Taylor's formula is

$$H^{(3)} - \sum_{j=1}^{s} p_j \sqrt{-1} \left(u_j \frac{\partial K^*}{\partial u_j} - v_j \frac{\partial K^*}{\partial v_j} \right)$$

where $H^{(3)}$ designates the third degree terms in H, and the dashes over the letters are omitted.

Evidently then the homogeneous cubic polynomial K^* can be so deter-

mined, and uniquely, that all of the third degree terms disappear in H. In fact a term in K^*

$$qu_1^{l_1}v_1^{m_1} \cdots u_s^{l_s}v_s^{m_s}$$

leads to a similar new term in $\bar{H}$ with coefficient

$$-q\left[\sum_{j=1}^{s} p_j\sqrt{-1}\,(l_j - m_j)\right],$$

where the term in brackets is not zero for

$$l_1 + m_1 + \cdots + l_s + m_s = 3.$$

This is in accordance with our statement since there can be no third degree term in the products $u_1v_1, \cdots, u_sv_s$.

Another transformation will lead to a modified fourth degree term in H of the same type,

$$H^{(4)} - \sum_{j=1}^{s} p_j\sqrt{-1}\left(u_j\frac{\partial K^*}{\partial u_j} - v_j\frac{\partial K^*}{\partial v_j}\right)$$

where K^* is now the arbitrary homogeneous polynomial of the fourth degree at our disposal. By the aid of it we can eliminate all the fourth degree terms in $H^{(4)}$ except those for which $l_i = m_i$, i. e. the very terms we do not attempt to remove.

By continuing in this way, we are led to the conclusion:

For the general Hamiltonian equations (1) *with first order stability, H can be taken in the form* $H(u_1v_1, \cdots, u_sv_s)$. *Inasmuch as the Hamiltonian equations then take the form for* $i = 1, \cdots s$,

$$\frac{du_i}{dt} = \frac{\partial H}{\partial \pi_i}\,u_i, \qquad \frac{dv_i}{dt} = -\frac{\partial H}{\partial \pi_i}\,v_i \qquad (\pi_i = u_iv_i)$$

of the completely prepared type (12), *it is clear that first order stability necessitates complete stability in the case of the Hamiltonian equations.*

Of course it is well known that the Hamiltonian equations admit a complete formal trigonometric solution in the case of first order stability.

We can now readily attack the question as to the possibility of reducing a completely stable system (1) to Hamiltonian form by means of a transformation belonging to the restricted formal group.

It was proved in the preceding section that the most general transformation of the formal group which kept preserved the completely prepared form (12) was given by (11). It was furthermore seen that $M_1, \cdots, M_s$ are not

altered, i. e. that $\overline{M}_i$ is obtained directly from M_i by substitution. But such a substitution changes the variables in $M_1, \cdots, M_s$ from $\pi_1, \cdots, \pi_s$ to $\overline{\pi}_1, \cdots, \overline{\pi}_s$ in accordance with the formulas

$$\pi_i = \overline{\pi}_i \bar{F}_i^* \bar{G}_i^*.$$

Imagine now the particular transformation performed which reduces the completely stable prepared equations (1) to Hamiltonian form, if that be possible. Since $M_1, \cdots, M_s$ are not altered, there must obtain

$$M_i = \partial H / \partial \overline{\pi}_i,$$

or, more briefly,

$$dH = \sum_{i=1}^{s} M_i d\overline{\pi}_i.$$

A necessary condition that such a transformation to Hamiltonian form be possible is therefore that

$$\sum_{j=1}^{s} M_j(\pi_1, \cdots, \pi_s) d\overline{\pi}_j$$

becomes an exact differential for suitable real power series

$$\overline{\pi}_i = \pi_i f_i(\pi_1, \cdots, \pi_s) \qquad (i = 1, \cdots, s)$$

where the constant terms c_i in f_i are not zero.

In particular this requires that the first degree terms in

$$\sum_{j=1}^{s} c_j M_j d\pi_j$$

yield an exact differential. More precisely if this term in M_i is

$$a_{i1}\pi_1 + \cdots + a_{is}\pi_s$$

we must have

$$\frac{a_{ij}}{a_{ji}} = \frac{c_j}{c_i} \qquad (i, j = 1, \cdots, s)$$

which is not in general true for $s > 2$.

The special case first pointed out when transformation to Hamiltonian form is possible, namely when $M_i - p_i\sqrt{-1}$ is divisible by $\xi_i\eta_i$, falls under the type treated with $d\overline{\pi}_i = dM_i/\sqrt{-1}$.

The necessary condition stated is also sufficient for transformation to Hamiltonian form.

In fact choose F_i^*, G_i^* in (11) so that $\overline{\pi}_i = \xi_i\overline{\eta}_i$, which is obviously

possible in an infinite variety of ways. Then $M_1, \cdots, M_s$ are unaltered, and also

$$M_1 d\bar{\pi}_1 + \cdots + M_s d\bar{\pi}_s = dH.$$

It follows that

$$M_i = \frac{\partial H}{\partial \bar{\pi}_i} \qquad (i = 1, \cdots, s)$$

which is what we desire to prove.

It is apparent in this way that while completely stable systems are of equal generality with the Hamiltonian type in a very wide formal sense, they are more general if only transformations of restricted type are admitted.

We turn now to the consideration of the general Pfaffian type of equation derived from the non-specialized variation problem.

At the equilibrium point the equations of variation of a Pfaffian system (15) are evidently of the form

$$\sum_{j=1}^{n} \left(a_{ij} \frac{dx_j}{dt} + \beta_{ij} x_j \right) = 0$$

where a_{ij} is skew-symmetric (i. e. $a_{ij} = -a_{ji}$) and β_{ij} is symmetric (i. e. $\beta_{ij} = \beta_{ji}$). These are linear differential equations with constant coefficients, the nature of whose solutions depends on the roots of the determinant equation in λ

$$| a_{ij}\lambda_j + \beta_{ij} | = 0.$$

In the case of stability the roots must be zero or pure imaginary, of course, while the elementary divisions of this λ matrix must be simple. We shall assume that there are no linear homogeneous combinations of these roots with integral coefficients which vanish. This is the general case.

In order to obtain a preliminary prepared form for the Pfaffian equations, we note first that the constant terms in P_i and Q can be omitted since they do not enter into the differential equations. Moreover, if the origin is an equilibrium point, there are no linear terms in Q. According to the principle of variation, we may directly transform the part of the integral I containing the linear terms so as to obtain the linear terms with the new variables. But it is known that by linear transformation and omission of an exact differential, the linear terms in P_i and P_j become x_j and $-x_i$, where x_i and x_j are conjugate variables in the sense of the Hamiltonian equations, while simultaneously Q takes the form

$$Q = \Sigma\, p_i (x_i^2 + x_j^2) + \cdots,$$

where only the lowest degree terms are written. Now, if we note that $-x_j dx_i$ differs from $x_i dx_j$ by a perfect differential we can write the lowest degree terms under the integral sign as

$$2\sum x_i dx_j - \sum p_j(x_i^2 + x_j^2)dt$$

where the index i runs over s values while j takes the conjugate index values. But these are essentially the first order terms in the Hamiltonian integral.

At this stage we may introduce the conjugate variables $\xi_i\eta_i$ by setting

$$\xi_i = x_i + \sqrt{-1}\,x_j, \quad \eta_i = x_i - \sqrt{-1}\,x_j\,,$$

where we find the linear terms above take essentially the form

$$\sum_{j=1}^{s} \xi_j d\eta_j,$$

while

$$Q = \sum_{j=1}^{s} p_j\sqrt{-1}\,\xi_j\eta_j + \cdots.$$

It is this preparation of the Pfaffian system in its lowest degree terms that we shall assume to be made, and we shall seek to reduce it further to complete Hamiltonian form by a succession of changes of variables

$$\xi_i = \bar{\xi}_i + \bar{F}_i, \quad \eta_i = \bar{\eta}_i + \bar{G}_i \qquad (i = 1, \cdots, s)$$

where $\bar{F}_i$, $\bar{G}_i$ are conjugate series in $\bar{\xi}_1, \cdots, \bar{\eta}_s$ beginning with terms of at least the second degree.

Let us write the prepared linear differential form as

$$R_1 d\xi_1 + \cdots + R_s d\xi_s + S_1 d\eta_1 + \cdots + S_s d\eta_s + Q dt,$$

and let us seek to eliminate the terms of second degree in $R_1, \cdots, R_s$ by a change of variables of the above type in which $\bar{F}_i$, $\bar{G}_i$ are arbitrary homogeneous quadratic polynomials.

We find immediately that the above linear form is preserved in character while we have

$$\bar{R}_i^{(2)} = R_i^{(2)} + \sum_{j=1}^{s} \xi_j \frac{\partial G_j}{\partial \xi_i} \qquad (i = 1, \cdots, s)$$

if $\bar{R}_i^{(2)}$, $R_i^{(2)}$ denote the second degree terms in $\bar{R}_i$ and R_i respectively. Evidently, by the subtraction of an exact differential, we can replace the sums on the right by the more abbreviated expressions $-G_i$.

Hence there exists a set of G_i which removes the remaining terms in $R_i^{(2)}$, and it is clear at the same time that in this way higher and higher degree terms in R_i, S_i can be disposed of by taking F_i, G_i as homogeneous polynomials of corresponding higher and higher degree. Thus a formal limiting composite transformation is set up which reduces $R_1, \cdots, R_s$ to zero, while taking $S_1, \cdots, S_s$ into $\xi_1, \cdots, \xi_s$ respectively, as desired.

In the general case of first order stability the Pfaffian equations can be transformed formally to Hamiltonian form.

§ 8. *Questions of Convergence.*

The methods which we have been following are based on the use of certain series. Their convergence or divergence is immaterial for the properties which we have so far discussed. Nevertheless the questions of convergence are of great importance for the theoretic treatment of a given dynamical problem.

For example, the Hamiltonian and Pfaffian types of equations (1) admit of the convergent series integral $H = \text{const.}$ and $P = \text{const.}$ respectively, whence in the case of first order stability we can immediately infer the *permanent* complete stability in the sense (A). Or again, there may exist a convergent series P for which

$$\partial/\partial x_1 (PX_1) + \cdots + \partial/\partial x_n (PX_n) = 0$$

in which case $\int P dv$ is an invariant volume integral (dv, element of volume); from the existence of this invariant volume integral, various important properties of recurrence can be deduced.

In the most favorable circumstances the normal form of the equations (12) can be obtained by ordinary transformations defined by convergent series. Here the series M_i and the trigonometric series solutions of the original equations will converge. The precise character of the motion can then be specified for all time. Concerning this integrable case we prove the following simple result:

For integrability of the system (1) *it suffices that the formal integrals*

$$\phi_i = \xi_i \eta_i = \text{const.} \qquad (i = 1, \cdots, s)$$

are given by convergent series.

In fact suppose that these real series do converge. Since they are formally factorable into conjugate factors $\xi_i \eta_i$, the separate components are given

by convergent series of course. This follows from Weierstrass's preparation theorem. Hence

$$M_i = \frac{1}{\xi_i}\frac{d\xi_i}{dt} = 1/\xi_i \sum_{j=1}^{n} \frac{\partial \xi_i}{\partial x_j} X_j \qquad (i = 1, \cdots, s)$$

are series given as a sum of formal quotients of two convergent series and so themselves are convergent. This demonstrates the complete integrability of (1) in case the series converge.

The existence of 'small divisors' in the series ξ_i, η_i shows that in general there will be no such convergent series ϕ_i, and the series M_i appear to be divergent for the same reason.

The possibilities of classification of completely stable systems (1) on the basis of diverse convergence properties are limitless. Every system of series or of differential equations which may be attached to (1) so as to have invariantive significance under the general formal group evidently affords a means of classification of systems (1) in which convergence or divergence plays a rôle.

It is with these few remarks that we terminate our reference to convergence questions. While on the purely formal side any completely stable system of equations (1) is to be regarded as equivalent to the extended Pfaffian or Hamiltonian type derived from a variational principle, it is clear that these more special types possess at least one ordinary integral and other convergence properties. In this non-formal direction the complete characterization of the completely stable equations (1) which may be reduced to the classical types still remains to be made.

Quaternary Quadratic Forms Representing all Integers.

By L. E. Dickson.

1. For the case in which only squares of the four variables occur, the forms representing all positive integers were determined by Ramanujan.* It is a much more difficult problem to determine the forms which involve also products of the variables whose coefficients may be odd. This paper develops a method of solving this problem completely.

The fact that one of these new forms represents all positive integers is equivalent to the fact that a related form involving only squares shall represent all positive multiples of a fixed integer, such as 4, 8, 12, 28, 44. A very few of the latter facts were known, and their importance has been emphasized in the literature.

The first part of the paper proves the necessary auxiliary theorems on ternary forms. They are useful also in other problems in the theory of numbers.

Part 1. Representation by Ternary Quadratic Forms.

2. Dirichlet † gave the simplest proof of the fact that $x^2+y^2+z^2$ represents all positive integers except those of the form $4^k(8n+7)$, but represents no integer of that form. This will be expressed briefly as

(1) $\qquad x^2+y^2+z^2$ represents all $\neq 4^k(8n+7)$.

From (1) we readily deduce ‡

(2) $\qquad x^2+2y^2+2z^2$ represents all $\neq 4^k(8n+7)$;

(3) $\qquad x^2+2y^2+4z^2$ represents all $\neq 4^k(16n+14)$.

* *Proc. Cambridge Phil. Soc.*, 19, 1916-9, pp. 11-21. Proofs of the needed theorems on ternary forms are given in the writer's paper cited in § 2.

† *Journal für Mathematik*, Vol. 40 (1850), p. 228; *Werke*, II, p. 91.

‡ Dickson, *Bulletin of the American Mathematical Society*, Vol. 33 (1927).

The proof is similar to that now given for

(4) $\qquad f = x^2 + 2y^2 + 8z^2$ represents all $\neq 8n + 5$ and $\neq 4^k(8n + 7)$.

If f is even, $x = 2X$, $f = 2F$, $F = 2X^2 + y^2 + 4z^2$. Applying (3), we conclude that f represents all even integers except those of the form $4^k(8n + 7)$. If f is odd, $x^2 \equiv 1$, $2y^2 \equiv 0$ or $2 \pmod 8$, whence $f \equiv 1$ or $3 \pmod 8$. But when $m \equiv 1$ or $3 \pmod 8$, (2) shows that m is represented by $x^2 + 2y^2 + 2z^2$. Then $x^2 \equiv 1$, $2(y^2 + z^2) \equiv 0$ or $2 \pmod 8$, whence y and z are not both odd. Permuting them if necessary, we may assume that $z = 2Z$, whence $m = x^2 + 2y^2 + 8Z^2$.

The writer proved (*loc. cit.*)

(5) $\qquad g = x^2 + y^2 + 3z^2$ represents all $\neq 9^k(9n + 6)$.

If g represents $3m$, x and y are multiples of 3, $x = 3X$, $y = 3Y$, and m is represented by $3X^2 + 3Y^2 + z^2$. The converse is true since the product of the latter by 3 is of the form g.

(6) $\qquad 3x^2 + 3y^2 + z^2$ represents all $\neq 9^k(3n + 2)$.

3. We shall need the following results:

(7) $\qquad \lambda = x^2 + xy + y^2 + z^2$ represents all $\neq 9^k(9n + 6)$;

(8) $\qquad \mu = x^2 + xy + y^2 + 3z^2$ represents all $\neq 9^k(3n + 2)$;

(9) $\qquad \rho = x^2 + y^2 + 2z^2 + xz$ represents all $\neq 49^k(49n + 7e)$, $e = 3, 5, 6$;

(10) $\qquad \sigma = x^2 + 2y^2 + 7z^2 - xy$ represents all $\neq 49^k(7n + e)$, $e = 3, 5, 6$.

By the Hessian H of a quadratic form whose cross products have even coefficients we shall mean the determinant of the halves of its second partial derivatives. By Eisenstein's* table of improperly primitive reduced positive ternary quadratic forms of Hessian $\leqq 100$, 2λ, 2μ, 2ρ, 2σ are the only reduced forms with all coefficients even of Hessians 6, 18, 14, 98, respectively.

We employ the form, lacking a term in xy,

(11) $\qquad 2Ax^2 + 2By^2 + 2Cz^2 + 2Ryz + 2xz.$

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* *Journal für Mathematik*, Vol. 41 (1851), pp. 186-9.

Its Hessian will be $2D$ if

$$B = A\Delta - D, \qquad \Delta = 4BC - R^2. \tag{12}$$

LEMMA 1. *If $p = 3$ or 7 and A is any positive integer not divisible by p, $2A$ is represented by the unique reduced form 2λ or 2ρ of Hessian $2p$ with all coefficients even.*

Here $D = p$. We take $\Delta = 4d$, $d = 2pt + 1$, and choose t so that $B = 4dA - p = 8Apt + 4A - p$ is a prime. This choice is possible in infinitely many ways, since the coefficients of the linear function of t are relatively prime. Since $B \equiv -p \equiv 1 \pmod 4$,

$$(-\Delta/B) = (d/B) = (B/d) = (-p/d) = (d/p) = (1/p) = 1.$$

Hence the last equation (12) is satisfied by integers C and $R = 2r$. Thus (11) is equivalent to 2λ or 2ρ. Hence the latter also represent $2A$.

LEMMA 2. *If $p = 3$ or 7 and A is any positive quadratic residue of p, $2A$ is represented by the unique reduced form 2μ or 2σ of Hessian $2p^2$ with all coefficients even.*

Here $D = p^2$. Take $\Delta = 8pl$, where l is prime to $2p$. By (12), $B = pb$, $b = 8lA - p$. Then b is a prime for infinitely many choices of l. Since $b \equiv -p \equiv 1 \pmod 4$,

$$(l/b) = (b/l) = (-p/l) = (l/p), \; (p/b) = (b/p) = (2lA/p),$$
$$(-2/b) = (2/b) = (2/p), \quad (A/p) = 1,$$
$$(-\Delta/b) = (-2/b)\,(p/b)\,(l/b) = 1.$$

To prove (7), note that, by Lemma 1, λ represents every positive integer prime to 3. Since $\lambda \equiv (x-y)^2 + z^2 \pmod 3$, λ is divisible by 3 if and only if $x - y$ and z are divisible by 3. Write $y = x + 3w$, $z = 3Z$. Then $\lambda = 3L$, $L = x^2 + 3xw + 3w^2 + 3Z^2$. Since $L \equiv 0$ or $1 \pmod 3$, λ represents no number of the form $3(3n+2)$. In L replace x by $X - Y$ and w by Y. We get μ written in capital letters. By Lemma 2, μ represents every positive quadratic residue $3n+1$ of 3. Thus λ represents every $9n+3$ and hence its product by 9^k. Finally, if $L \equiv 0$, then $x \equiv 0 \pmod 3$, and x, y, z are all multiples of 3, whence λ is the product of a like form by 9. This proves (7).

Next, (8) follows from (7), since we proved that, when $3m$ is represented by λ, m is represented by L and hence by μ. The converse is true since *

$$3(x^2+xy+y^2) = X^2+XY+Y^2, \quad X=x-y, \quad Y=x+2y.$$

To prove (9), note that, by Lemma 1, ρ represents every positive integer prime to 7. Since $\rho \equiv (x-3z)^2+y^2 \pmod 7$, ρ is divisible by 7 if and only if $x=3z+7X$, $y=7Y$. Then $\rho=7R$, $R=2z^2+7zX+7X^2+7Y^2$. Hence ρ represents none of the integers excluded in (9). For $z=Z-2X$, R becomes $r=2Z^2-ZX+X^2+7Y^2$, which is of type σ in (10). By Lemma 2, σ and hence r represents all positive quadratic residues $7n+1, 2, 4$ of 7. Hence ρ represents their products by $7\cdot 49^k$. If $R\equiv 0$, then $z\equiv x \equiv y \equiv 0 \pmod 7$, and ρ is the product of a like form by 49.

Finally, (10) follows from (9), since we proved that, when $7m$ is represented by ρ, m is represented by R and hence by σ. The converse is true since †

$$7(x^2+2y^2-xy) = X^2+2Y^2-XY, \quad X=x-4y, \quad Y=2x-y.$$

The following results are not required in this paper. Lemma 1 holds also if $p=5$ or 13. In the proof take $d=2pt+\nu$, where ν is odd and a quadratic non-residue of p. Lemma 2 holds also if $p=5$ and A is any non-residue of 5. As before, we prove that

(13) $x^2+y^2+2z^2+xy+xz+yz$ represents all $\neq 25^k(25n\pm 5)$;

(14) $2x^2+2y^2+2z^2-xy-xz-yz$ represents all $\neq 25^k(5n\pm 1)$.

These forms are the only reduced ones whose doubles have the Hessian 10 and 50 respectively.

4. *Forms of Hessian* 6. The reduced forms are

$$\tau = x^2+y^2+6z^2, \quad \omega = x^2+2y^2+3z^2, \quad 2\lambda.$$

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* For if ω is an imaginary cube root of unity, the product of the factor $x-\omega y$ of x^2+xy+y^2 by the corresponding factor $1-\omega$ of 3 is $X-\omega Y$. Also, (8) follows directly from Lemmas 1 and 2. If μ is divisible by 3, $x=y+3X$; write $y=Y-X$. Then μ becomes the triple of a form λ.

† For, if $\epsilon^2+2-\epsilon=0$, the product of the factor $x-\epsilon y$ by the corresponding factor $1-2\epsilon$ of $7=1+2\cdot 2^2-2$ is $X-\epsilon Y$. Also, (10) follows from Lemmas 1 and 2. If σ and hence $(x+3y)^2$ is divisible by 7, $y=2x+7Z$ and $\sigma=7S$, $S=x^2+7xZ+14Z^2+z^2$. For $x=X-3Z$, $z=Y$, S becomes ρ written in capital letters.

Suppose that $\omega \equiv 10 \pmod{16}$. Then $\omega \equiv 2 \pmod 4$. Thus $\epsilon = x^2 + 3z^2$ is even. Whether x and z are both even or both odd, $\epsilon \equiv 0 \pmod 4$. Hence y is odd and $10 \equiv \omega \equiv \epsilon + 2$, $\epsilon \equiv 8 \pmod{16}$. If x and z are odd, $\epsilon \equiv 4 \pmod 8$. Hence $x = 2X$, $z = 2Z$, $X^2 + 3Z \equiv 2 \mod 4$), contrary to the above. Thus ω represents * no integer of the form $2(8n + 5)$. This is used in proving

(15) $\quad \tau = x^2 + y^2 + 6z^2$ represents all $\neq 9^k(9n + 3)$.

If τ represents $2m$, then $x + y = 2X$, $x - y = 2Y$, $m = X^2 + Y^2 + 3z^2$. Conversely, the latter implies $2m = (X + Y)^2 + (X - Y)^2 + 6z^2$. Hence by (5) the even integers represented by τ are the doubles of the integers represented by g and include all $\neq 9^k(9m + 3)$, where $m = 2n + 1$. We next prove that τ represents every positive odd integer a not divisible by 3. We employ

$$\kappa = ax^2 + by^2 + cz^2 + 2ryz + 2xz,$$

whose Hessian is $H = a\Delta - b$, where $\Delta = bc - r^2$. Take $H = 6$, $\Delta = 16k$, $k \equiv 1 \pmod 6$. Then $b = 16ka - 6 = 2\beta$, where $\beta = 8ka - 3$ is a prime for infinitely many choices of k. Thus b is not represented by ω. Since b and the odd a are represented by κ, κ is equivalent to neither ω nor 2λ and hence is equivalent to τ. Since

$$(-\Delta/\beta) = (-k/\beta) = (\beta/k) = (-3/k) = +1,$$

$\Delta = bc - r^2$ is satisfied by integers c and r.

If τ is divisible by 3, $x = 3X$, $y = 3Y$, $\tau = 3T$, $T = 3X^2 + 3Y^2 + 2z^2$, whence τ represents no $3(3n + 1)$.

It remains only to prove that τ represents $a = 3A$, where A is a positive odd integer $3n - 1$. We employ

$$F = ax^2 + 2By^2 + cz^2 + 4Ryz + 2sxz,$$
$$s = 1 + A, \quad B = 8kA - 3, \quad k \equiv 1 \pmod 6,$$

with k chosen so that B is a prime. The Hessian of F is 6 if

$$B + 2AB + A^2B + 3 - 3ABc + 6AR^2 = 0.$$

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* It was proved (*Bulletin*, l. c.) that a positive integer is represented by ω if and only if it is not of the form $4^k(16n + 10)$.

Replace the first term B by its value and cancel A; we get

$$8k + 6R^2 - BP = 0, \quad P = 3c - A - 2.$$

Hence $8k(1+P) \equiv 0$, $P \equiv -1 \pmod 3$. Hence c is an integer. The congruence $48k + (6R)^2 \equiv 0 \pmod B$ holds if and only if

$$(-48k/B) = (-3k/B) = +1.$$

The latter follows from

$$(k/B) = (B/k) = (-3/k) = (k/3),$$
$$(-3/B) = (B/3) = (8kA/3) = (k/3).$$

5. *Forms of Hessian* 12. We shall prove that

(16) $\delta = 2x^2 + 2y^2 + 3z^2$ represents all $\neq 8n+1$ and $\neq 9^k(9n+6)$.

If δ is even, it is the double of a form (15). Henceforth, let z be odd. Evidently $\delta \not\equiv 1 \pmod 8$. Let a be a positive integer such that $a \equiv 3 \pmod 4$, $a \neq 9^k(9n+6)$. Then a is represented by g in (5). If z is even, $g \equiv 0$, 1 or 2 (mod 4), contrary to $a \equiv 3$. Hence z is odd, $x^2 + y^2 \equiv 0 \pmod 4$, $x = 2X$, $y = 2Y$. Hence a is represented by $2(X+Y)^2 + 2(X-Y)^2 + 3z^2$ and therefore by δ.

It remains only to prove that δ represents every positive integer b such that $b \equiv 5 \pmod 8$, $b \neq 9^k(9n+6)$. Then b is represented by g in (5). First, let z be odd. Then $x^2 + y^2 \equiv 2 \pmod 4$, whence x and y are odd. Thus $x+y = 2X$, $x-y = 2Y$, and $g = 2X^2 + 2Y^2 + 3z^2$, so that b is represented by δ. Second, let $z = 2\zeta$. Thus $x+y$ is odd. Since we may permute x and y, we may take $x = 2\xi$, y odd. Then $5 \equiv g \equiv 4\xi^2 + 1 + 12\zeta^2 \pmod 8$, and $\xi + \zeta$ is odd. Thus

(17) $$X = \tfrac{1}{2}(x - 3z), \quad Z = \tfrac{1}{2}(x+z)$$

are odd integers and

(18) $$X^2 + 3Z^2 = x^2 + 3z^2.$$

Hence we have a representation of b by g in which the unknowns are now all odd, as in the first case.

We shall need the fact that

(19) $\epsilon = x^2 + 3y^2 + 4z^2$ represents all $\neq 4n+2$ and $\neq 9^k(9n+6)$.

If ϵ is even, $x = y + 2X$, $\epsilon = 4\lambda$, where λ is of the form (7), and conversely. Thus (19) holds for even integers. Next, let a be a positive odd integer $\neq 9^k(9n+6)$. Then a is represented by g in (5). Then $x+y+z$ is odd. First, let two of x, y, z be even. Permuting x and y if necessary, we may take x even. Then $g = 4X^2 + y^2 + 3z^2$, whence a is represented by ϵ. Next, let x, y, z be all odd. Changing the sign of x or z if necessary, we may take $x \equiv 1$, $z \equiv -1 \pmod 4$. Then X and Y in (17) are even integers for which (18) holds. Hence the second case reduces to the first.

6. *Corollaries.* We require the results

(20) $\quad a = 3x^2 + 4y^2 + 12z^2$ represents all $\neq 4n+1, 4n+2, 9^k(12n+11), 9^k(12n+8)$;

(21) $\quad \beta = x^2 + 3y^2 + 12z^2$ represents all $\neq 4n+2, \quad 9^k(3n+2)$.

If m is represented by β, $3m$ is represented by $3x^2 + Y^2 + 4Z^2$ with $Y = 3y$, $Z = 3z$, and conversely, whence (19) implies (21).

If a is even, $x = 2X$, a is the quadruple of a form (6). Next, let a and hence x be odd. Then $a \equiv 3 \pmod 4$. Any integer represented by a is represented by β with x even. Conversely, if $\beta \equiv 3 \pmod 4$, y is odd and x is even, whence β is of the form a. This proves (20). Finally,

(22) $\quad \gamma = x^2 + 12y^2 + 12z^2$
represents all $\neq 4n+2, 4n+3, 9^k(12n+5), 9^k(12n+8)$;

(23) $\quad \nu = 3x^2 + 4y^2 + 4z^2$
represents all $\neq 4n+1, 4n+2, 9^k(36n+15), 9^k(36n+24)$.

If ν is even, ν is the quadruple of a form of type (5). Next, let ν be odd. Then $\nu \equiv 3 \pmod 4$. Write $Y = y + z$, $Z = y - z$. Then ν becomes $\delta = 3x^2 + 2Y^2 + 2Z^2$. Conversely, if $\delta \equiv 3 \pmod 4$, x is odd and $Y + Z$ is even $= 2y$. Write $Y - Z = 2z$. Then δ becomes ν. Applying (16), we get (23).

If γ represents m, then $3m$ is represented by ν with y and z multiples of 3. Conversely, the latter follows if $\nu = 3m$, whence $\gamma = m$. Thus (23) implies (22).

PART 2. QUATERNARY QUADRATIC FORMS.

7. Let τ be a positive ternary quadratic form such that $f = u^2 + \tau$ represents all positive integers. Let m be the minimum positive integer repre-

sented by τ. If $m \geqq 3$, f would not represent 2. Postponing the case $m = 2$, let $m = 1$. Let H be the Hessian of the form 2τ all of whose coefficients are even. Since the minimum of 2τ is 2, a known theorem * states that 2τ is equivalent to a form $2T$ such that $4T - X^2$ is a reduced form $C = Ly^2 + 2Myz + Nz^2$ of Hessian $2H$, where $X = 2x + ty + sz$, while the values of t and s are both chosen from 0, 1. Further, we may assume † that $M \geqq 0$ and omit the case $L = 4$, $M = 2$, $s = 0$. Since the coefficients of $X^2 + C = 4T$ are all multiples of 4, we have

$$(24) \qquad L + t^2 \equiv N + s^2 \equiv 0 \pmod 4, \quad M + ts \equiv 0 \pmod 2.$$

The Hessian $2H = LN - M^2$ of C is therefore a multiple of 4, whence H is even, $H = 2h$. By hypothesis, $f = u^2 + T$ represents all positive integers. Hence

$$F = 4f = 4u^2 + X^2 + C$$

represents all positive multiples of 4. Since C is reduced, its minimum > 0 is L. If $L > 12$, $F = 12$ therefore requires that $C = 0$. Since 12 is not a sum of two squares, this proves that $L \leqq 12$. Since $t = 0$ or 1, (24) gives $L \equiv 0$ or $-1 \pmod 4$. Hence the only possible values of L are 4, 8, 12, 3, 7, 11. The least positive multiple of 4 not represented by $4u^2 + X^2 + Ly^2$ is $l = 28, 56, 24, 24, 84, 88$ for the corresponding value $4, \cdots, 11$ of L. Since F represents l, we have $C \leqq l$ for certain integers y, z, $z \neq 0$. Then

$$(25) \qquad LC \equiv (Ly + Mz)^2 + 4hz^2$$

implies that

$$(26) \qquad 4h \leqq Ll.$$

Hence there is a limited number of values of h and hence of reduced forms C. The number of forms T is therefore finite.

8. The simplest case is $L = 4$. Since C is reduced, $2M \leqq L$, $M = 0, 1, 2$. By (24), $t = 0$, $M \equiv 0 \pmod 2$, and $N + s^2$ is divisible by 4.

First, let $s = 0$. If $M = 2$, our ternary form is known to be equivalent to one with $s = 1$. Hence there remains here the case $M = 0$. By (26),

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* Landau, *Handbuch ... Verteilung der Primzahlen*, 1909, p. 545.

† Eisenstein, *Journal für Mathematik*, 41, 1851, 141-90.

$h \leqq 28$. Since $h = N$ is a multiple $4m$ of 4, $1 \leqq m \leqq 7$, and $f = u^2 + x^2 + y^2 + mz^2$ represents every positive integer (Ramanujan).

Second, let $s = 1$. We may write $N = 4m - 1$. There are two cases.

(i) $M = 0$. Then $h = N$, $h \leqq 28$, $1 \leqq m \leqq 7$,

$$(27_1) \qquad 4f = 4u^2 + X^2 + 4y^2 + Nz^2, \quad X = 2x + z \equiv z \pmod 2.$$

This is of the form $F = U^2 + X^2 + Y^2 + Nz^2$. Conversely, if F is a multiple of 4, either z is even and U, X, Y are all even, or z is odd and just one of them is odd. Permuting them, we may take X odd, whence U and Y are even. In both cases, $U = 2u$, $Y = 2y$, $X \equiv z \pmod 2$, and we have (27_1). To show that F represents all multiples of 4, we apply (1). Let $a = 4^k(8n + 7)$ be a multiple of 4, whence $k \geqq 1$. For $z = 2^{k-1}$, $a - Nz^2 = 4^{k-1}P$, where $P = 32n + 28 - N \equiv 1 \pmod 4$. Thus P is positive and a sum of three squares.

(ii) $M = 2$. Then $h = N - 1$, $h \leqq 28$, $1 \leqq m \leqq 7$,

$$(27_2) \qquad 4f = U^2 + X^2 + W^2 + (4m - 2)z^2, \quad U = 2u,$$
$$X = 2x + z, \quad W = 2y + z.$$

Conversely, if this form is a multiple of 4, either z, U, X, W are all even, or z is odd, and just one of U, X, W is even. Permuting them, we may take U even, $X \equiv z$, $W \equiv z \pmod 2$, and deduce (27_2). To show that it represents all multiples of 4, take $z = 2^{k-1}$. Then

$$a - (4m - 2)z^2 = 4^{k-1}Q, \quad Q = 32n + 28 - 4m + 2 \equiv 2 \pmod 4, \; Q > 0.$$

Hence Q is a sum of three squares.

THEOREM 1. *For $1 \leqq m \leqq 7$, $f_1 = u^2 + y^2 + x^2 + xz + mz^2$ and $f_2 = f_1 + yz$ both represent all integers. Expressed otherwise, $X^2 + Y^2 + U^2 + (4m - 1)z^2$ and $X^2 + W^2 + U^2 + (4m - 2)z^2$ both represent all positive multiples of* 4.

9. *The case $L = 8$.* By (24), $t = 0$, M is even, and $N = 4k - s$. Write $M = 2\mu$. By $2M \leqq L$, $\mu \leqq 2$. We postpone to § 10 the case $\mu = 1$. Here $\mu = 2\nu$, $\nu = 0$ or 1. Then $4f = F$,

$$(28) \qquad F = \phi + \sigma z^2, \quad \phi \equiv 4u^2 + X^2 + 2W^2, \quad \sigma = 4k - s - 2v^2,$$

where $X = 2x + sz$, $W = 2y + vz$. Thus

$$(29) \qquad X \equiv sz, \quad W \equiv vz \pmod 2.$$

Conversely, if F represents a multiple of 4, (29) hold. Evidently $X \equiv \sigma z \equiv sz \pmod 2$. If z is even, X and W are even. If z is odd, $s^2 + 2W^2 + \sigma \equiv 0$, $2W^2 - 2v^2 \equiv 0 \pmod 4$, whence $W \equiv v \equiv vz \pmod 2$.

We have $8N - M^2 = 4h \leqq 8 \cdot 56$ by (26), whence $\sigma = N - 2v^2 \leqq 56$. Also, $N \geqq L$. Hence $2 \leqq k \leqq 14$. By 3, ϕ represents exclusively all positive integers except $4^e(16n + 14)$. If the latter is a multiple of 4, it is $a = 4^p(64n + 56)$. For $z = 2^p$, F represents a if ϕ represents

$$a - 4^p\sigma = 4^p q, \quad q = 64n + 56 - \sigma \geqq 0.$$

If $s = 1$, σ and q are odd, and $4^p q$ is represented by ϕ. If $s = v = 0$, (28) gives $f = u^2 + x^2 + 2y^2 + kz^2$, which represents all positive integers (Ramanujan). Finally, let $s = 0$, $v = 1$. Then $q \equiv 14 \pmod{16}$ only when $k \equiv -1 \pmod 4$, whence $k = 3, 7, 11$; $\sigma = 10, 26, 42$, respectively. Except in these cases $4^p q$ is represented by ϕ. For $z = 2^{p+1}$,

$$B = a - \sigma z^2 = 4^{p+1} b, \quad b = 16n + 14 - \sigma.$$

If $\sigma = 10$, $b = 4(4n + 1)$, and B is represented by ϕ. If $\sigma = 26$, $b = 16(n - 1) + 4$ is the quadruple of an odd integer; if $n > 0$, B is represented by ϕ. For $\sigma = 26$, $n = 0$, $a = 4^p \cdot 56$. If $p > 0$, take $z = 2^{p-1}$; then $a - \sigma z^2$ is the product of 4^{p-1} by $198 \equiv 6 \pmod{16}$. Hence unless a is 56, it is represented by F, while 56 is not. Finally, let $\sigma = 42$. Then $b = 16(n - 2) + 4 > 0$ if $n > 1$. For $n = 0$ or 1 and $p > 0$, take $z = 2^{p-1}$; then $a - \sigma z^2$ is the product of 4^{p-1} by 182 or 438, each $\equiv 6 \pmod{16}$. Hence unless a is 56 or $64 + 56$, it is represented by F, while they are not.

THEOREM 2. *For* $2 \leqq k \leqq 14$, $s = 0$ *or* 1, $v = 0$ *or* 1,

$$f = u^2 + x^2 + sxz + 2y^2 + 2vyz + kz^2$$

represents all positive integers except 14 *when* $k = 7$ *and* 14 *and* 30 *when* $k = 11$, *with* $s = 0$, $v = 1$ *for all exceptions. Expressed otherwise, the form* (28) *with only square terms represents all positive multiples of* 4 *except* 56 *and* 120 *in the cases mentioned.*

10. *The case* $L=8$, $M=2$. Now $8f=G$,

$$G=\psi+hz^2,\quad \psi=2X^2+W^2+8u^2,\quad h=2N-1,\quad N=4k-s, \tag{30}$$

where $X=2x+sz$, $W=4y+z$. Hence

$$X\equiv sz \pmod 2,\quad W\equiv z \pmod 4. \tag{31}$$

Conversely, the latter follow if G represents a multiple of 8. Evidently $W\equiv z \pmod 2$. If z is odd, $2X^2+1+h\equiv 0 \pmod 4$, whence X^2+N is even and $X\equiv s\equiv sz \pmod 2$. If $W-z=4r+2$, $W+z$ is divisible by 4; after changing the sign of z if necessary, $W-z$ is divisible by 4. Next, let z be even, $z=2\zeta$. Then W and X are even, $W=2\omega$. Then

$$G\equiv 4(\omega^2+h\zeta^2)\equiv 0 \pmod 8,\quad \omega\equiv\zeta \pmod 2.$$

Thus (31) hold.

By (4), any multiple of 8 not represented by ψ is of the form $a=4^e\cdot 16(8n+7)$. Take $z=2^e$. Since $h\leqq 112$ by (26), $a-hz^2$ is the product of 4^e by a positive integer $\equiv 2s+1\equiv 1$ or $3 \pmod 8$, and hence is represented by ψ.

THEOREM 3. *For* $2\leqq s\leqq 14$, $s=0$ *or* 1,

$$f=u^2+x^2+sxz+2y^2+yz+kz^2$$

represents all positive integers. In other words, (30) *represents all positive multiples of* 8.

11. *The case* $L=3$. Then $M=0$ or 1. By (24), $t=1$, $M=s$, $N=4k-s$. Then $12f=F$,

$$F=\phi+4hz^2,\quad \phi=3U^2+3X^2+W^2,\quad h=3k-s, \tag{32}$$

where $U=2u$, $X=2x+y+sz$, $W=3y+sz$. Conversely, if F represents a multiple of 12, we may write

$$U\equiv 0,\quad X\equiv W \pmod 2,\quad W\equiv sz \pmod 3.$$

By (6), any multiple of 12 not represented by ϕ is of the form $a=9^e\cdot 9(12r+8)$. By (26), $h\leqq 18$. Take $z=3^e$. Then $a-4hz^2$ is the product of 9^e by $g=9\cdot 12r+72-4h\geqq 0$. Thus $g\equiv s \pmod 3$. If

$s=1$, $G=9^e g$ is represented by ϕ. Next, let $s=0$. If k is not divisible by 3, g is divisible by 3, but not by 9, whence G is represented by ϕ. For $k=3$, $g=9(12r+4)$ and G is represented by ϕ. For $k=6$, $g=9\cdot 12r$ and G is repreesnted by ϕ if r is not divisible by 3. When r is divisible by 3, take $z=2\cdot 3^e$, whence $a-4hz^2=9^{e+1}\cdot 12(r-2)$ is represented by ϕ.

THEOREM 4. *For* $1\leqq k\leqq 6$, $s=0$ *or* 1,

$$u^2+x^2+xy+sxz+y^2+syz+kz^2$$

represents all positive integers. In other words, the form (32) *represents all positive multiples of* 12.

12. *L odd.* By (24), $t=1$, $N=4k-s$, and $M+s$ is even. Write

$$U=2u,\quad X=2x+y+sz,\quad W=Ly+Mz.$$

Using (25), we get

$$K=4Lf=LU^2+LX^2+W^2+4hz^2, \tag{33}$$

$$U\equiv 0,\quad W\equiv X \pmod 2,\quad W\equiv Mz \pmod L. \tag{34}$$

Conversely, let K be divisible by $4L$. Since $4h=LN-M^2$, $W^2-M^2z^2\equiv 0 \pmod L$. Changing the sign of z if necessary, and noting that $L=3$, 7, or 11 is a prime, we get (34_3). Since $L\equiv -1 \pmod 4$, $K\equiv 0$ requires that either W, U, X are all even or that W is odd and just one of U and X is even. Permuting the latter if necessary, we may take U even, X odd. In both cases the first two congruences (34) hold. Hence f represents all positive integers if and only if K represents all positive multiples of $4L$. Write

$$W=X+2\xi,\quad L=4\lambda-1.$$

Then we may remove the factor 4 from (33) and get

$$F=Lf=\phi+hz^2,\quad \phi=Lu^2+\lambda X^2+X\xi+\xi^2. \tag{35}$$

Conversely, if F is divisible by L,

$$(X+2\xi)^2\equiv M^2z^2 \pmod L,\quad X+2\xi=Mz+Ly=W,$$

after changing the sign of z if necessary. Hence f represents all positive integers if and only if F represents all positive multiples of L. For $L=3$,

ϕ is of type (8) and we may give a new proof of Theorem 4 by use of (35).

We have

$$h = Lk - \tau, \quad \tau = \tfrac{1}{4}(Ls + M^2), \quad 0 \leqq M \leqq \tfrac{1}{2}L. \tag{36}$$

13. *The case* $L = 7$. Since ϕ is of type (10), any multiple of 7 not represented by ϕ is of the form $a = 49^e \cdot 49(7n + \nu)$, where $\nu = 3$, 5, or 6 is a non-residue of 7. By (26), $h \leqq 7 \cdot 21$. By (36), $h = 7k - \tau$, where $\tau = 0$, 1, 2, or 4. Thus $2 \leqq k \leqq 21$. Take $z = 7^e$. Then

$$a - hz^2 = G = 49^e g, \quad g = 49(7n + \nu) - 7k + \tau \geqq 0.$$

Hence G is represented by ϕ if $\tau \neq 0$, or if $\tau = 0$ and k is prime to 7. Henceforth, let $\tau = 0$, $k = 7\kappa$, $\kappa = 1$, 2, or 3. Then $G = 49^{e+1}\gamma$, where $\gamma = 7n + \nu - \kappa$. Next, take $z = 2 \cdot 7^e$. Then

$$a - hz^2 = D = 49^{e+1}\delta, \quad \delta = 7n + \nu - 4\kappa.$$

If $\nu = 6$, $\gamma = 7n + 2$ for $\kappa = 2$, $\delta = 7n + 2$ for $\kappa = 1$,
$\delta = 7(n-1) + 1$ for $\kappa = 3$.

If $\nu = 5$, $\gamma = 7n + 4$ for $\kappa = 1$, $\gamma = 7n + 2$ for $\kappa = 3$,
$\delta = 7(n-1) + 4$ for $\kappa = 2$.

If $\nu = 3$, $\delta = 7n + 2$ or $7n + 1$ for $\kappa = 1$ or 2.

In all these cases, G or D is represented by ϕ if it is positive. It is negative only when $n = 0$, $\nu = 6$, $\kappa = 3$, or $\nu = 5$, $\kappa = 2$. If $e > 0$ in the former case, $z = 7^{e-1}$ gives $a - hz^2 = 49^e(49 \cdot 6 - 3)$, which is represented by ϕ. But if $e = 0$, $a = 49 \cdot 6$ is not represented by ϕ, since $a - hz^2 = 49(6 - 3z^2)$ is positive only when $z^2 = 0$ or 1 and then the second factor is a non-residue of 7. In the second of the exceptional cases, $a = 5 \cdot 49^{e+1}$, $h = 2 \cdot 49$. We may take $z = \epsilon 7^m$, ϵ prime to 7. According as $m \geqq e$ or $m < e$,

$$a - hz^2 = 49^{e+1}(5 - 2\epsilon^2 49^{m-e}), \quad 49^{m+1}(5 \cdot 49^{e-m} - 2\epsilon^2),$$

and the second factors are non-residues of 7 if positive.

Finally, let $\nu = \kappa = 3$. Then G is represented by ϕ unless n is a positive multiple of 7. In that case, take $z = 3 \cdot 7^e$. Then

$$a - hz^2 = 49^{e+1}m, \quad m = 7n + 3 - 27 = 7(n-4) + 4 > 0,$$

and the former is represented by ϕ.

THEOREM 5. *For* $2 \leqq k \leqq 21$; $s = 0, 1$; $M = 0, 1, 2, 3$; $M + s$ *even,*

$$u^2 + x^2 + xy + sxz + 2y^2 + \tfrac{1}{2}(M + s)yz + ks^2$$

represents all positive integers except 42 *when* $k = 21$, $s = M = 0$, *and* $35 \cdot 49^e$ *when* $k = 14$, $s = M = 0$. *In other words, for* h *in* (36) *and* $L = 7$, *the form* (33) *represents all positive multiples of* 28 *except* $28 \cdot 42$ *and* $28 \cdot 35 \cdot 49^e$ *in the cases mentioned.*

14. *The case* $L = 11$. Then ϕ in (35) represents all positive integers except $121^k(11n + \nu)$, where ν ranges over the non-residues 2, 6, 7, 8, 10 of 11. Let it be a multiple $a = 121^e \cdot 121\ (11n + \nu)$ of 11. By (36), $h = 11k - \tau$, where τ is 0 or one of the quadratic residues 1, 3, 4, 5, 9 of 11. By (26), $h \leqq 2 \cdot 11^2$. Thus $3 \leqq k \leqq 22$. Take $z = 11^e$. Then

$$a - hz^2 = G = 121^e g, \quad g = 121(11n + \nu) - 11k + \tau \geqq 0.$$

Thus G is represented by ϕ if $\tau \neq 0$, or if $\tau = 0$ and k is prime to 11. Henceforth, let $\tau = 0$, $k = 11\kappa$, $\kappa = 1$ or 2. Then $G = 121^{e+1}\gamma$, $\gamma = 11n + \nu - \kappa$. Next, take $z = 2 \cdot 11^e$. Then

$$a - hz^2 = D = 121^{e+1}\delta, \quad \delta = 11n + \nu - 4\kappa.$$

First, let $\kappa = 1$. For $\nu = 2$ or 6, $\gamma = 11n + 4$ or $11n + 5$. For $\nu = 7$ or 8, $\delta = 11n + 3$ or $11n + 4$. Hence G or D is represented by ϕ.

Next, let $\kappa = 2$. For $\nu = 6$ or 7, $\gamma = 11n + 1$ or $11n + 5$. For $\nu = 2$, $\gamma = 11n$ and G is represented by ϕ if $n = 0$, while if $n > 0$, $\delta = 11(n - 1) + 5 > 0$ and D is represented. Finally, let $\nu = 8$. If $n = 0$, $D = 0$.

If $n > 0$, take $z = 3 \cdot 11^e$. Then ϕ represents

$$a - hz^2 = 121^{e+1}[11(n - 1) + 1].$$

THEOREM 6. *For* $3 \leqq k \leqq 22$; $s = 0, 1$; $M = 0, 1, 2, 3, 4, 5$; $M + s$ *even,*

$$u^2 + x^2 + xy + sxz + 3y^2 + \tfrac{1}{2}(M + s)yz + kz^2$$

represents all positive integers. In other words, for h *in* (36), $L = 11$, *the form* (33) *represents all multiples of* 44.

15. *The case* $L = 12$. By (24), $t = 0$, $N = 4k - s$, $M = 2\mu$, $0 \leqq \mu \leqq 3$. Also, $h = 3N - \mu^2$.

(37) $\quad 12f = 12u^2 + 3X^2 + W^2 + hz^2, \quad X = 2x + sz, \quad W = 6y + \mu z.$

However this form is divisible by 4 also when z is even, and X and W are odd. Since the usual converse argument fails here, we apply devices appropriate to the necessary sub-cases.

First, let μ be even, $\mu = 2\nu$, whence $\nu = 0$ or 1. In (37), $W = 2Y$, $Y = 3y + \nu z$, whence

$$(38) \qquad F = 12f = a + hz^2, \quad a = 12u^2 + 3X^2 + 4Y^2, \quad X \equiv sz \pmod 2, \quad Y \equiv \nu z \pmod 3.$$

Conversely, the latter follow if F is divisible by 12.

We have $h = 12k - \tau$, where $\tau = 3s + 4\nu^2$ takes the values 0, 3, 4, 7. By (26) and $N \geqq L$, $3 \leqq k \leqq 6$. By (20), a represents all multiples of 12 except $a = 9^e \cdot 9(12n + 8)$. For $z = 3^e$,

$$D = a - hz^2 = 9^e d, \quad d = 9 \cdot 12n + 72 - 12k + \tau \geqq 0.$$

If $\tau = 7$, $d \equiv 7 \pmod{12}$, and D is represented by a.

Let $\tau = 3$. Then d is divisible by 9 only if $k \equiv 1 \pmod 3$, whence $k = 4$. If $k \neq 4$, $d \equiv 3 \pmod{12}$, and D is represented by a. If $k = 4$, $h = 45$, $d = 9(12n + 3)$, and D is represented unless $12n + 3$ is divisible by 9, whence $n = 3m + 2$. In the latter case, take $z = 2 \cdot 3^e$. Then a represents

$$a - hz^2 = 9^{e+1} \cdot 12(3m + 1).$$

If $\tau = 4$, $d \equiv 4 \pmod{12}$, and D is represented by a.

Let $\tau = 0$. If k is prime to 3, d is divisible by 12, but not by 9, when a represents D. If $k = 3$, $d = 9(12n + 4)$, and D is represented. Finally, let $k = 6$. Then $D = 9^{e+1} \cdot 12n$ is represented unless $n = 3m$, $m > 0$. Take $z = 2 \cdot 3^e$. Then a represents

$$a - hz^2 = 9^{e+1} \cdot 12(3m - 2) > 0.$$

THEOREM 7. *For $k = 3, 4, 5, 6$; $s = 0, 1$; $\nu = 0, 1$.*

$$u^2 + x^2 + sxz + 3y^2 + 2\nu yz + kz^2$$

represents all positive integers. In other words, (38) *represents all positive multiples of* 12.

16. $L = 12$, *μ odd*, $s = 0$. By (37),

$$(39) \qquad 12f = F = \gamma + hz^2, \quad \gamma = 12u^2 + 12x^2 + W^2, \quad h = 12k - \mu^2,$$

where $W = 6y + \mu z$. Conversely, if F is a multiple of 12, we may write $W \equiv \mu z \pmod 6$. Since $N = 4k \geqq 12$, $h \leqq 3 \cdot 24$, $k = 3, 4, 5,$ or 6. By (22), γ represents all multiples of 12 except $a = 9^e \cdot 9(12n + 8)$. For $z = 3^e$,

$$D = a - hz^2 = 9^e d, \quad d = 9(12n + 8) - 12k + \mu^2 \geqq 0.$$

If $\mu = 1$, then $d \equiv 1 \pmod{12}$, and γ represents D.

Let $\mu = 3$. If $k = 4$ or 5, then $d \equiv 9 \pmod{12}$ and d is not divisible by 9, whence γ represents D. If $k = 6$, then $d = 9(12n + 1)$ and γ represents D. If $k = 3$, $e > 0$, take $z = 3^{e-1}$; then γ represents $a - hz^2 = 9^e g$, where $g = 9(12n + 8) - 3 \equiv 9 \pmod{12}$. There remains only the case $\mu = k = 3$, $e = 0$. Then $a = 9(12n + 8)$ is not represented by F. For, $a - hz^2 = 9\delta$, $\delta = 12n + 8 - 3z^2$. According as z is odd or even, $\delta \equiv 5$ or $8 \pmod{12}$, whence 9δ is not represented by γ.

THEOREM 8. *For $k = 3, 4, 5, 6$; $\mu = 1, 3$, $u^2 + x^2 + 3y^2 + \mu yz + kz^2$ represents all positive integers except those of the form $9n + 6$ when $k = \mu = 3$. In other words,* (39) *represents all positive multiples of* 12 *except* $12(9n + 6)$ *when* $k = \mu = 3$.

17. $L = 12$, $\mu = s = 1$. Then $4k - 1 = N \geqq 12$, $k \geqq 4$, $h = 12k - 4 \leqq 3 \cdot 24$, $k \leqq 6$. Hence

$$f = u^2 + x^2 + xz + 3y^2 + yz + kz^2 \qquad (k = 4, 5, 6)\ldots \tag{40}$$

We first determine the integers represented by f when z is even, $z = 2Z$. Write $X = x + Z$, $Y = 3y + Z$. Then $3f = F$,

$$F = \phi + hZ^2, \quad \phi = 3u^2 + 3X^2 + Y^2, \quad Y \equiv Z \pmod 3. \tag{41}$$

Conversely, $F \equiv 0$ implies $Y^2 - Z^2 \equiv 0 \pmod 3$. Changing the sign of Z if necessary, we have $Y \equiv Z$. We shall prove that F represents all positive multiples of 3 except 18, and, when $k = 5$ or 6, except 45. By (6), ϕ represents exclusively all positive integers except $9^k(3n + 2)$. If the latter be a multiple of 3, it is $a = 9^e \cdot 9(3n + 2)$. If $e > 0$, take $Z = 3^{e-1}$; then

$$a - hZ^2 = D = 9^{e-1}d, \quad d = 9^2(3n + 2) - h > 0, \quad d \equiv 1 \pmod 3.$$

Hence ϕ represents D, and F represents a if $e > 0$. Next, let $e = 0$. Take $Z = 1$. Then $a - hZ^2 = 27n + 18 - h \equiv 1 \pmod 3$ and is positive if $n > 0$ when $k = 4$, $n > 1$ when $k = 5$ or 6.

For the exceptional cases we employ (40) with $y=0$, $z=1$, whence $f=u^2+x^2+x+k$. We have

$f=6$ if	k	u	x
	4	0	1
	5	1	0
	6	0	0

$f=15$ if	k	u	x
	5	2	2
	6	3	0

THEOREM 9. *The forms* (40) *represent all positive integers.*

This and Theorem 10 are equivalent to the fact that (37) subject to the condition $X\equiv z \pmod 2$ represents all multiples of 12 apart from the exceptions noted.

18. $L=12$, $s=1$, $\mu=3$. Then $h=12(k-1)\leqq 3\cdot 24$, $k\leqq 7$. Thus

$$f=u^2+x^2+xz+3y^2+3yz+kz^2 \qquad (k=4,5,6,7). \tag{42}$$

We first determine the integers represented by f when $z=2Z$. Write $X=x+Z$, $Y=y+Z$. Then f becomes

$$F=\phi+mZ^2, \quad \phi=u^2+X^2+3Y^2, \quad m=4(k-1). \tag{43}$$

By (5), ϕ represents exclusively all positive integers except $a=9^e(9n+6)$. For $Z=3^e$, $D=a-mZ^2=9^e d$, $d=9n+6-m$. When $k=4, 5, 6$, $d\equiv 3, 8, 4 \pmod 9$ and D is represented by ϕ unless it be negative, i. e., unless $n=0$ for $k=4$, $n=0$ or 1 for $k=5, 6$. Next, for $e>0$, take $Z=3^{e-1}$. Then

$$G=a-mZ^2=9^{e-1}g, \quad g=9(9n+6)-m>0.$$

When $k=5, 6, 7$, $g\equiv 2, 7, 3 \pmod 9$, and G is represented by ϕ.

It remains to consider only the following cases:

(i) $k=4$, $a=9^e\cdot 6$. Then * $f=a$ for $x=z=3^e$, $u=y=0$.

(ii) $k=5$, $a=6$ or 15. Neither a is represented by F, since $m=16$. For $y=0$, $z=1$, $f=u^2+x^2+x+5$. Then $f=6$ if $u=1$, $x=0$; $f=15$ if $u=x=2$.

(iii) $k=6$, $a=6$ or 15. Neither a is represented by F, since $m=20$. For $x=y=0$, $z=1$, $f=u^2+6=6$ or 15 if $u=0$ or 3.

* But F does not represent any such a. For, if $Z=3^m\epsilon$, ϵ prime to 3, $P=a-12Z^2$ is negative if $m\geqq e$. If $m<e$, $P=9^m p$, $p=9^{e-m}\cdot 6-12\epsilon^2\equiv 6 \pmod 9$.

(iv) $k=7$, $a=9n+6$. For $Z=1$, $a-24Z^2=9(n-2)$ is represented by ϕ if $n\geqq 2$ and $n-2\not\equiv 0$ or 6 (mod 9) and hence if $n\not\equiv 2$ or 8 (mod 9). Similarly, for $Z=2$, $a-24Z^2=9(n-10)$ is represented by ϕ if $n\geqq 10$ and $n\not\equiv 1$ or 7 (mod 9). Hence a is represented by F unless $n=0$, 1, or 8; whence $a=6$, 15, or 78. We enquire whether $12a$ is represented by (37), viz., $B=\beta+72z^2$, with $X-z$ even, where $\beta=12u^2+3X^2+W^2$.

By (21), $\beta\not\equiv 72$, whence $z^2=1$, $X=0$, and $X-z$ is odd. Thus (42) does not represent 6. Next, $B=12\cdot 15$ for $u=0$, $z=1$, $X=3$, $W=9$, whence $f=15$ for $u=0$, $x=y=z=1$. Finally, $B=12\cdot 78$ for $u=3$, $z=1$, $X=3$, $W=27$, whence $f=78$ for $u=3$, $x=z=1$, $y=4$.

THEOREM 10. *The forms* (42) *represent all positive integers with the exception of* 6 *when* $k=7$.

19. By way of summary, we note that Theorems 1-10 give 329 forms which represent all positive integers, and six forms which represent all but one or two small integers. We have not included the known forms u^2+T which represent all positive integers, where T denotes

$$x^2+y^2+sz^2\,(s\leqq 7),\qquad x^2+2y^2+sz^2\;(2\leqq s\leqq 14),$$
$$x^2+3y^2+sz^2\;(s=3,\,4,\,5,\,6).$$

Including the latter 24, we obtain

THEOREM 11. *There are exactly* 353 *classes of positive ternary forms* T *whose minimum is* 1 *such that* u^2+T *is a quaternary form representing all positive integers.*

Similar methods apply in the remaining case $m=2$ (§ 7), and the number of reduced forms T is again finite. The results will be published on another occasion.

Reduction Formulas for the Number of Representations of Integers in Certain Quadratic Forms.

By E. T. Bell.

1. *Introduction.* We shall illustrate by its application to three interesting formulas, summarized in § 3, an algebraic method whereby, knowing the number of representations of an arbitrary integer in a given quadratic form in $r \geqq 2$ indeterminates, we can obtain a finite reduction formula for the number of representations in a quadratic form in rs indeterminates, $s > 1$. The formulas in any case can be obtained by strictly arithmetical methods depending on a simple use of the theory of binary quadratic forms of negative determinant $-c$; I prefer however to use an algebraic method, partly on account of its independent interest, but chiefly because such a method can be more easily extended.

Let $c,\ n,\ p > 0$ be integers, of which p is prime; write $(r,\ s)$ for the G.C.D. of $r,\ s,$ and $N\ [n = f]$ for the number of representations of n in the form f. The three formulas mentioned refer to

$$f \equiv x^2 + cy^2 + p(z^2 + ct^2),$$

so that $N[n = f]$ is the total number of sets of integers $x,\ y,\ z,\ t \geqq 0$ such that, for n fixed,

$$n = x^2 + cy^2 + p(z^2 + ct^2).$$

Any integer $t > 0$ is of the form $p^a n$, where $(p,\ n) = 1,\ a \geqq 0$. We shall call the expression of $N[p^a n = f],\ a > 0$, as an algebraic function of $N[n = f]$ the *reduction* of $N[p^a n = f]$, and similarly for any form at least one of whose coefficients is divisible by p.

For $c = 1,\ 2,\ 3$ the complete reductions of $N[p^a n = f]$ were stated by Liouville,* who however suppressed all details concerning the only cases in each instance which are not immediately obvious. His interesting and useful formulas therefore stand in need of proof, the more so as a direct use of them

* *Journal des Mathématiques*, Vol. 10 (1865), pp. 43, 49; Vol. 11, p. 211. In § 3 here we considerably shorten Liouville's statements of his results; also, in §§ 6-8 we derive formulas which unify several of his.

greatly abridges Pepin's memoir * on Liouville's numerous theorems concerning special quadratic forms which also were published without proof.

A particular interest attaches to those reduction formulas which express $N[p^a n = f]$, $a > 0$, in terms of $N[n = f]$ and algebraic functions of the divisors of the given integers n, p, c. Liouville's theorems for $c = 1, 2, 3$ are of this type; equally simple results can be found for $c = 5, 7$ and n properly restricted, also in any case where there is but one class of binary quadratic forms of determinant $-c$, or where each order contains but one class. This will appear from the algebra for $c = 1, 2, 3$, although the details are given only for these values of c.

2. *A difference equation.* Let $(m, 2) = 1$, $n = 2^s m$, $s \geqq 0$, $(n, p) = 1$, and write $\sigma(j) =$ the sum of all the divisors of j. Then

$$\sigma(2^s p^a m) = \sigma(p^a)\sigma(n), \quad a \geqq 0.$$

Now if $N[p^a n = f] = \psi(a)\phi(n)$, where ϕ is independent of p, a, and for p fixed $\psi(a)$ depends only on a, we need attend only to $\psi(a)$ in the determination of $N[p^a n = f]$. The difference equation for $\psi(a)$ in all cases with which we shall be concerned is of the type

$$\psi(a+1) + \psi(a) = k\sigma(p^a), \tag{1}$$

where k is independent of p, a. The solution of this equation is

$$\psi(a) = k \sum_{j=0}^{a-1} (-1)^j \sigma(p^{a-j-1}) + (-1)^a \psi(0),$$

and hence, since $\sigma(p^r) = (p^{r+1} - 1)/(p-1)$, we have

$$\psi(a) = (-1)^a \psi(0) + \frac{k}{p-1}\left[\frac{p\{p^a - (-1)^a\}}{p+1} - \tfrac{1}{2}\{1 - (-1)^a\}\right], \tag{2}$$

or, as it may be written,

$$\psi(a) = \frac{k(2p^{a+1} - p - 1)}{2(p^2-1)} + (-1)^a\left[\psi(0) - \frac{k}{2(p+1)}\right]. \tag{3}$$

For m, n, s as stated, k is of the form $\chi(s)\sigma(m)$, where $\chi(s)$ is a function of s alone.

3. *Summaries for $c = 1, 2, 3$.* Write

$$f \equiv x^2 + y^2 + p(z^2 + t^2), \tag{4}$$

$$g \equiv x^2 + 2y^2 + p(z^2 + 2t^2), \tag{5}$$

$$h \equiv x^2 + 3y^2 + p(z^2 + 3t^2), \tag{6}$$

* Ibid., Vol. 6 (1890), p. 1.

for p prime, and let m, n, s be as in § 2.

Then, for the form f,

$$N[2^a m = f] = c\, N[m = f], \tag{4.1}$$

where $c = 2$, 6 according as $a = 1$ or $a > 1$;

$$p \equiv -1 \bmod 4, \quad N[p^a n = f] = N[n = f], \quad a \geqq 0; \tag{4.2}$$

$$p \equiv 1 \bmod 4, \quad N[p^a n = f] = \psi(a), \tag{4.3}$$

where $\psi(a)$ is as in (2), (3), with $k = 16\sigma(m)$ or $48\sigma(m)$ according as $s = 0$ or $s > 0$.

For the form g, with $p > 2$ (the case $p = 2$ is treated in § 5), we have

$$p \equiv 5, 7 \bmod 8, \quad N[p^a n = g] = N[n = g], \quad a \geqq 0; \tag{5.1}$$

$$p \equiv 1, 3 \bmod 8, \quad N[p^a n = g] = \psi(a), \tag{5.2}$$

where $\psi(a)$ is as before, with $k = 8\sigma(m)$, $16\sigma(m)$, $48\sigma(m)$ according as $s = 0$, $s = 1$, $s > 1$.

For the form h,

$$p \equiv 1 \bmod 6, \quad N[p^a n = h] = N[n = h], \quad a \geqq 0; \tag{6.1}$$

$$p \equiv -1 \bmod 6, \quad (p, 3) = (p, n) = (3, n) = 1, \quad N[p^a 3^b n = f] = N[p^a n = f], \quad b \geqq 0; \tag{6.2}$$

$$p \equiv -1 \bmod 6, \quad (n, 3p) = (m, 2) = 1, \quad n = 2^s m, \quad N[p^a n = h] = \psi(a), \tag{6.3}$$

with $k = 8\sigma(m)$, $8(2^{s+1} - 3)\sigma(m)$ according as $s = 0$, $s > 0$.

The enumerations (4.1) — (6.3) are evidently exhaustive for the respective forms; the only cases offering any difficulty are

$$(4.3),\ (5.2),\ (6.3), \tag{7}$$

to which we return in §§ 6-8.

4. *Connection of f, g, h with principal binary quadratic forms of determinant $-c$.* In order that n may be represented in the form $x^2 + cy^2$ of

negative determinant $-c$, it is necessary that $(-c \mid n) = 1$, where (as always henceforth) $(r \mid s)$ is the Jacobi-Legendre symbol. Hence, in order that n be not representable in $x^2 + cy^2$ it is sufficient that $(-c \mid n) = -1$.

Consider all the representations of $p^a n$, where p, a, n are as before, in the form

$$F \equiv x^2 + cy^2 + p(z^2 + ct^2).$$

If $(-c \mid p) = -1$, so that p is not representable in $x^2 + cy^2$, then $N[p^a n = F] \neq 0$, $a > 0$, only when $x \equiv y \equiv 0 \bmod p$. Put therefore $x = p\xi$, $y = p\eta$; then

$$N[p^a n = x^2 + cy^2 + p(z^2 + ct^2)] = N[p^{a-1} n = z^2 + ct^2 + p(\xi^2 + c\eta^2)],$$

whence, repeating the reduction a sufficient number of times we have

$$N[p^a n = F] = N[n = F], \quad a \geqq 0.$$

Either this or a similar argument disposes of all cases in § 3 except (7), and the like can obviously be extended to any form $f = f_1 + p(f_2 + \cdots + f_s)$, where

$$f_j = \alpha x_j^2 + \beta x_j y_j + \gamma y_j^2 \quad (j = 1, \cdots, s),$$

and, with obvious modifications, to any forms f_j all of the same kind in different sets of indeterminates.

5. *The Algebraic Method.* To illustrate the simpler essentials of the algebraic method of finding theorems of the type § 3(7), we shall apply it to the form

$$\phi \equiv x^2 + y^2 + 2z^2 + 2t^2,$$

verifying first that

$$N[2^{a+1} m = \phi] = N[2^a m = x^2 + y^2 + z^2 + t^2],$$

where $(m, 2) = 1$ as before. Multiply within the second bracket by 2, in order to make the arguments $2^{a+1}m$, $2^a m$ agree, so that like powers of a parameter q in the generating functions of both N's may be compared:

$$N[2^{a+1} m = \phi] = N[2^{a+1} m = 2x^2 + 2y^2 + 2z^2 + 2t^2].$$

The respective generators are

$$\theta_3^4(q^2) = \Sigma\, q^{2n} N[2n = 2x^2 + 2y^2 + 2z^2 + 2t^2],$$

where Σ refers to $n = 0, 1, 2, \cdots$, and the modification of

$$\theta_3^2(q)\theta_3^2(q^2) = \Sigma\, q^n N[n = x^2 + y^2 + 2z^2 + 2t^2],$$

from which are selected only those q^n such that $n \equiv 0 \bmod 2$. This is accomplished by changing q to $-q$ and adding;

$$\theta_3^2(q)\theta_3^2(q^2) + \theta_0^2(q)\theta_3^2(q^2) = 2\Sigma\, q^{2n}\ N[2n = \phi].$$

Hence it must be shown that

$$\theta_3^2(q)\theta_3^2(q^2) + \theta_0^2(q)\theta_3^2(q^2) = 2\theta_3^4(q^2),$$

or, what is the same,

$$\theta_3^2(q) + \theta_0^2(q) = 2\theta_3^2(q^2),$$

which is correct, by the transformation of the second order. The verification is therefore complete.

The particular point to be noted is that we select only those q^n in the second generator for which $n \equiv 0 \bmod p$, where p ($= 2$ in this case) is the prime concerned in the given form ϕ. This generalizes to any p.

The foregoing however is a mere verification, and we are interested in finding such reduction formulas *ab initio*. For this the order of the steps is reversed. We start from any known theorem concerning numbers of representations in binary quadratic forms, translate it into its theta equivalent, multiply the result throughout by theta constants to obtain a new identity which, when read as a relation between N's, yields a theorem concerning numbers of representations. The appropriate theorems for binary forms will be found in Dirichlet's *Zahlentheorie,* or in the papers of Pepin and others abstracted in Volume 3 of Dickson's *History.*

6. *Form* $f \equiv x^2 + y^2 + p(z^2 + t^2)$. The appropriate binary theorem is

$$N[n = x^2 + y^2] = 4\xi(n),$$

where $\xi(n) =$ the excess of the number of $4t + 1$ divisors of n over the number of $4t + 3$ divisors; $\xi(r)\xi(s) = \xi(rs)$ if $(r, s) = 1$, and if $p \equiv 1 \bmod 4$

(p prime as always), $\xi(p^a) = a + 1$. Since we need attend only to $p \equiv 1 \bmod 4$ for f, we take only this case, and have, in the notation of § 2,

$$N[p^a n = x^2 + y^2] + N[p^{a-2} n = x^2 + y^2] = 2N[p^{a-1} n = x^2 + y^2]$$

as the simplest difference equation satisfied by ξ. The translation of this into an identity between generators is (coefficients of q^j, $j = p^a n$, are compared),

$$\Sigma\, q^{pn} N[pn = x^2 + y^2] + \Sigma\, q^{p^2 n}\, N[n = x^2 + y^2] = 2\Sigma\, q^{pn}\, N[n = x^2 + y^2],$$

in which Σ refers to $n = 0, 1, 2, \cdots$. We now replace this by its θ-equivalent; the first Σ is the only one requiring any attention. To restate this Σ as a function of θ's we must select from

$$\theta_3^2(q) = \Sigma\, q^n\, N[n = x^2 + y^2]$$

only those terms for which the exponent of q is a multiple of p. Let r be a primitive (algebraic) pth root of unity. Then

$$\sum_{j=0}^{p-1} \theta_3^2(r^j q) = p\, \Sigma\, q^{pn}\, N[pn = x^2 + y^2],$$

and hence the required θ-equivalent is

$$\sum_{j=0}^{p-1} \theta_3^2(r^j q) + p\theta_3^2(q^{p^2}) = 2p\, \theta_3^2\, (q^p),$$

which is the analytic expression of our linear difference equation for N. Hence any consequence of this θ-identity will yield a true proposition. We wish here to derive from it a theorem concerning enumerations in quadratic forms in 4 indeterminates, and therefore we multiply the identity throughout by $\theta_3^2(q)$ or by $\theta_3^2(q^p)$, choosing the latter as it gives a simpler result. We have then

$$\theta_3^2(q^p) \sum_{j=0}^{p-1} \theta_3^2(r^j q) + p\theta_3^2(q^p)\, \theta_3^2(q^{p^2}) = 2p\theta_3^4(q^p).$$

From the way in which the Σ in this was obtained, only multiples of p will occur as exponents of q in the power series for the first product on the left. Equating coefficients of q we get

$$N[pn = x^2 + y^2 + p(z^2 + t^2)]$$
$$+ N[pn = (px)^2 + (py)^2 + p(z^2 + t^2)]$$
$$= 2N[pn = px^2 + py^2 + pz^2 + pt^2],$$
$$2N[n = x^2 + y^2 + z^2 + t^2], = 16[2 + (-1)^n]\sigma(m),$$

the last by Jacobi's theorem on 4 squares, where as always $n = 2^s m, (m, 2) = 1$, $s \geqq 0$.

Hence, in the notation of § 2 we have

$$\psi(a + 1) + \psi(a) = 16[2 + (-1)^n]\sigma(p^a m),$$

whose solution $\psi(a)$ is given by (3) with $k = 16[2 + (-1)^n]\sigma(m)$, and this is equivalent to (4.3).

In the foregoing analysis the use of pth roots of unity is not a necessity; we might have used throughout the full generator in its N form. But in more complicated situations of a similar kind it is advantageous to have all generators expressed in finite theta form on account of the transformations thus suggested.

7. *Form* $g \equiv x^2 + 2y^2 + p(z^2 + 2t^2)$. We start from the known N for the principal binary quadratic form of determinant -2,

$$N[2^s m = x^2 + 2y^2] = 2\Sigma\,(-2 \mid d) \equiv 2\omega(m),$$

where $(m, 2) = 1$, and Σ refers to all divisors d of m. From the properties of Jacobi's symbol it follows that if r, s are relatively prime integers, then $\omega(rs) = \omega(r)\omega(s)$. To obtain the initial difference equation for ω we need the explicit form of $\omega(p^\beta)$, $\beta > 0$ (p prime, as always). We have

$$\omega(p^\beta) = \sum_{j=0}^{\beta} (-2 \mid p^j) = \Sigma\,(-2 \mid p)^j,$$

and therefore

$$\omega(p^\beta) = \beta + 1, \quad \tfrac{1}{2}[1 + (-1)^\beta],$$

according as $(-2 \mid p) = 1$ or -1. Explicitly then

$$\omega(p^\beta) = \tfrac{1}{2}[1 + (-2 \mid p)]\,(\beta + 1)$$
$$+ \tfrac{1}{2}[1 - (-2 \mid p)] \cdot \tfrac{1}{2}[1 + (-1)^\beta],$$

so that

$$4\omega(p^\beta) = 2\beta + 3 + (-1)^\beta + (-2 \mid p)\,[2\beta + 1 - (-1)^\beta].$$

Hence, for $a > 0$,

$$4\omega(p^{a+1}) = 2a + 5 - (-1)^a + (-2 \mid p)\,[2a + 3 + (-1)^a],$$
$$4\omega(p^a) = 2a + 3 + (-1)^a + (-2 \mid p)\,[2a + 1 - (-1)^a],$$
$$4\omega(p^{a-1}) = 2a + 1 - (-1)^a + (-2 \mid p)\,[2a - 1 + (-1)^a];$$

whence we have the required difference equation

$$\omega(p^{a+1}) + \omega(p^{a-1}) = 2\omega(p^a) + (-1)^a\,[(-2 \mid p) - 1].$$

Referring to § 3 (7) we see that it is necessary for our present purpose to attend only to this equation in the case $(-2 \mid p) = 1$, so that

$$(-2 \mid p) = 1, \quad \omega(p^{a+1}) + \omega(p^{a-1}) = 2\omega(p^a), \quad a > 0.$$

Proceeding now exactly as in § 6, we write down from this

$$\sum q^{np} N[np = x^2 + 2y^2] + \sum q^{np^2} N[n = x^2 + 2y^2] = 2\sum q^{np}\ N[n = x^2 + 2y^2],$$

where Σ refers to $n = 0, 1, 2, \cdots$, and (as in all that follows) $(-2 \mid p) = 1$. Translating as before to theta functions, with $r =$ a primitive pth root of unity, we get

$$\sum_{j=0}^{p-1} \theta_3(r^j q)\theta_3(r^{2j} q^2) + p\theta_3(q^{p^2})\theta_3(q^{2p^2}) = 2p\theta_3(q^p)\theta_3(q^{2p}),$$

and therefore

$$\theta_3(q^p)\theta_3(q^{2p}) \sum_{j=0}^{p-1} \theta_3(r^j q)\theta_3(r^{2j} q) + p\theta_3(q^p)\theta_3(q^{2p})\theta_3(q^{p^2})\theta_3(q^{2p^2}) = 2p\theta_3^{\,2}(q^p)\theta_3^{\,2}(q^{2p}),$$

which steps up from binary to quaternary forms upon equating coefficients of like powers of q. With $(n, p) = 1$ we get

$$N[p^{a+1}n = g] + N[p^a n = g] = 2N[p^a n = x^2 + 2y^2 + z^2 + 2t^2].$$

The last N being a known function we can proceed. For $(2, m) = 1$,

$$N[2^s m = x^2 + y^2 + 2(z^2 + t^2)] = b\sigma(m),$$

where $b = 4, 8, 24$ according as $s = 0, 1, > 1$, which follows at once from § 5 and Jacobi's theorem on 4 squares. Hence for $n = 2^s m$, $s \geqq 0$, we have

$$b = 2\{1 - (-1)^n\} + 4\{2 + (-1)^{n/2}\}\{1 + (-1)^n\},$$

and therefore in the notation of § 2 (1)

$$\psi(a+1) + \psi(a) = k\sigma(p^a),$$

with here

$$k = 2b\sigma(m),$$

whose solution is given by § 2 (3) with this value of k. This is equivalent to (5.2).

8. *Form* $h \equiv x^2 + 3y^2 + p(z^2 + 3t^2)$. An even integer n is represented in $x^2 + 3y^2$ only if $n \equiv 0 \bmod 4$. From the known theory of binary forms we have

$$N[m = x^2 + 3y^2] = 2\sum (-3 \mid d) \equiv 2\delta(m), \quad (m, 2) = 1,$$
$$N[2^{2s} m = x^2 + 3y^2] = 6\delta(m), \quad s > 0,$$

the Σ referring to all divisors d of m. When $m \equiv 0 \bmod 3$ the Jacobi symbol $(-3 \mid 3d) = 0$. Proceeding as in § 7 we find

$$4\delta(p^\beta) = 2\beta + 3 + (-1)^\beta + (-3 \mid p)\,[2\beta + 1 - (-1)^\beta],$$

which gives the difference equation

$$\delta(p^{a+1}) + \delta(p^{a-1}) = 2\delta(p^a) + (-1)^a[(-3 \mid p) - 1],$$

for which, by § 2 (7), we need discuss only the case $p \equiv -1 \bmod 6$; whence

$$\delta(p^{a+1}) + \delta(p^{a-1}) = 2\delta(p^a).$$

As before, with $(n, p) = 1$, we find from the identity between products of 4 theta functions derived from this,

$$N[p^{a+1} n = h] + N[p^a n = h] = 2N[p^a n = x^2 + 3y^2 + z^2 + 3t^2].$$

The last N is well known; it may be obtained if desired analytically from

the transformation of the third order of the theta functions. It can be written

$$N[n = x^2 + 3y^2 + z^2 + 3t^2] = 4[\sigma(n) - \{1 - (-1)^n\}\ \sigma(m)],$$

where $n = 2^s m$, $s \geqq 0$, $(m, 2) = 1$. Hence in this case

$$\psi(a+1) + \psi(a) = k\sigma(p^a)$$

where

$$k = 4[1 - (-1)^n + \{1 + (-1)^n\}(2^{s+1} - 3)]\ \sigma(m),$$

and for this k we get (6.3) from (3).

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Generalization of Certain Theorems of Bohl, [Second Paper].

By F. H. MURRAY.

In a preceding communication with the same title * it was shown that a number of existence theorems of Bohl † can be generalized in such a manner as to apply to the solutions of a system of differential equations in the canonical form

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}, \quad i = 1, \cdots, n,$$

near a point solution $x_i = a_i$ $(i = 1, \cdots, n)$ when certain conditions are satisfied. In the present paper these conditions are replaced by less stringent ones; the methods of proof of certain existence theorems are very similar to those employed in the first paper, and these proofs are given here in an abbreviated form. In addition, the asymptotic properties of certain trajectories are discussed by an extension of the methods of Bohl.

On account of the more complicated form of certain quadratic forms which occur here, it has been convenient to leave undetermined certain constants which are determined explicitly in the first paper; this procedure, together with the algebraic method of transforming the canonical equations, reduces to a small amount the results common to both papers.

1. *Transformation of the canonical equations.*

Assume the function $F(x_1, \cdots, x_n; y_1, \cdots, y_n)$ to be continuous, together with its partial derivatives of the first and second orders, within a region

(D) $$|x_i| \leqq D, \quad |y_i| \leqq D, \quad i = 1, \cdots, n,$$

and suppose $\dfrac{\partial F}{\partial x_i} = \dfrac{\partial F}{\partial y_i} = 0$, $(i = 1, \cdots, n)$ if $x_k = y_k = 0$. The canon-

* *Amer. Journal of Math.*, vol. XLVII (1925), pp. 25-44.

† P. Bohl, "Ueber die Bewegungen eines mechanischen Systems, etc.", *Crelle's Journal*, Band 127 (1904), pp. 179-276.

ical equations

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}, \qquad i = 1, \cdots, n, \tag{1.01}$$

become

$$\begin{aligned} \frac{dx_i}{dt} &= \sum_{r=1}^{n} \frac{\partial^2 F}{\partial y_i \partial x_r} x_r + \sum_{r=1}^{n} \frac{\partial^2 F}{\partial y_i \partial y_r} y_r + F_i', \\ \frac{dy_i}{dt} &= -\sum_{r=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_r} x_r - \sum_{r=1}^{n} \frac{\partial^2 F}{\partial x_i \partial y_r} y_r + F_i'', \end{aligned} \qquad i = 1, \cdots, n, \tag{1.02}$$

and the equations of variation

$$\begin{aligned} \frac{d\xi_i}{dt} &= \sum_{r=1}^{n} \frac{\partial^2 F}{\partial y_i \partial x_r} \xi_r + \sum_{r=1}^{n} \frac{\partial^2 F}{\partial y_i \partial y_r} \bar{\eta}_r \\ \frac{d\bar{\eta}_i}{dt} &= -\sum_{r=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_r} \xi_r - \sum_{r=1}^{n} \frac{\partial^2 F}{\partial x_i \partial y_r} \bar{\eta}_r \end{aligned} \qquad i = 1, \cdots, n, \tag{1.03}$$

possess characteristic exponents which satisfy an equation with real coefficients

$$\Delta(a) = 0. \tag{1.04}$$

By hypothesis the functions F_k', F_k'' possess continuous derivatives of the first order which vanish with the variables x_i, y_i $(i, k = 1, \cdots, n)$.

From the classical theory of differential systems of the form (1.03) it is known that if a is a root of equation (1.04), the solutions can be associated in certain groups of the form

$$\begin{aligned} &\xi_i' = e^{at}\xi_i', && \bar{\eta}_i' = e^{at}\eta_i', \\ &\xi_i'' = e^{at}(\xi_i'' + t\xi_i'), && \bar{\eta}_i'' = e^{at}(\eta_i'' + t\eta_i'), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\xi_i^{(\nu)} = e^{at}\left(\xi_i^{(\nu)} + t\xi_i^{(\nu-1)} + \cdots + \frac{t^{\nu-1}}{\lfloor \nu - 1}\, \xi_i'\right), \\ &\bar{\eta}_i^{(\nu)} = e^{at}\left(\eta_i^{(\nu)} + t\eta_i^{(\nu-1)} + \cdots + \frac{t^{\nu-1}}{\lfloor \nu - 1}\, \eta_i'\right). \end{aligned} \tag{1.05}$$

Suppose another group of solutions represented by

$$\begin{aligned} &\bar{x}_i' = e^{a't}x_i', && \bar{y}_i' = e^{a't}y_i', \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\bar{x}_i^{(\mu)} = e^{a't}\left(x_i^{(\mu)} + \cdots + \frac{t^{\mu-1}}{\lfloor \mu - 1}\, x_i'\right), \\ &\bar{y}_i^{(\mu)} = e^{a't}\left(y_i^{(\mu)} + \cdots + \frac{t^{\mu-1}}{\lfloor \mu - 1}\, y_i'\right). \end{aligned} \tag{1.051}$$

Then from the relation which is satisfied by any pair of solutions of (1.03),

$$(1.06) \qquad \sum_{i=1}^{n} (\xi_i^{(r)} \bar{y}_i^{(s)} - \bar{\eta}_i^{(r)} \bar{x}_i^{(s)}) = C,$$

we obtain, if $a + a' \neq 0$,

$$\sum_{i=1}^{n} (\xi_i' y_i' - \eta_i' x_i') = 0,$$

and in general

$$\sum_{i=1}^{n} (\xi_i^{(r)} y_i^{(s)} - \eta_i^{(r)} x_i^{(s)}) + t \left[\sum_{i=1}^{n} (\xi_i^{(r-1)} y_i^{(s)} - \eta_i^{(r-1)} x_i^{(s)}) + \sum_{i=1}^{n} (\xi_i^{(r)} y_i^{(s-1)} - \eta_i^{(r)} x_i^{(s-1)})\right] + \cdots = 0,$$

whence

$$(1.07) \qquad \sum_{i=1}^{n} (\xi_i^{(r)} y_i^{(s)} - \eta_i^{(r)} x_i^{(s)}) = 0,$$
$$r = 1, 2, \cdots, \nu, \quad s = 1, 2, \cdots, \mu, \quad (r + s > 1).$$

If $a + a' = 0$, we obtain in the same manner

$$\sum_{i=1}^{n} (\xi_i^{(r)} \bar{y}_i^{(s)} - \bar{\eta}_i^{(r)} \bar{x}_i^{(s)}) = \sum_{i=1}^{n} (\xi_i^{(r)} y_i^{(s)} - \eta_i^{(r)} x_i^{(s)}) + t \left[\sum_{i=1}^{n} (\xi_i^{(r)} y_i^{(s-1)} - \eta_i^{(r)} x_i^{(s-1)}) + \sum_{i=1}^{n} (\xi_i^{(r-1)} y_i^{(s)} - \eta_i^{(r-1)} x_i^{(s)})\right] + \cdots = C',$$

from which further relations can be derived; but these will not be used in what follows.

Suppose that no exponent is zero, and that the function F is real for real values of its arguments. Then it is known that to each group of ν solutions of (1.03) corresponding to the exponent a there is a group of ν solutions with the exponent $-a$.

If a is one exponent, the conjugate imaginary $\bar{a}$ is another, and a fundamental set of solutions of (1.03) can be so chosen that the constants which appear in any group corresponding to $\bar{a}$ are the conjugate imaginaries of the corresponding constants appearing in the group with the exponent a.

It will be assumed that there is only one solution in each group corresponding to a pure imaginary exponent; the solutions corresponding to a pure imaginary exponent a_r, $\left(\frac{a_r}{\sqrt{-1}} > 0\right)$, can be normalized * in such a manner

* *Amer. Journal.*, *l. c.*, pp. 28-30.

that if (ξ_i, η_i), corresponding to a_r, and (x_i, y_i) corresponding to $-a_r$ are conjugate imaginary sets, then

$$(1.08) \qquad \sum_{i=1}^{n} (\xi_i y_i - \eta_i x_i) = \epsilon_r = \pm (1/a_r),$$

while if (x'_i, y'_i) is any other solution corresponding to $-a_r$,

$$(1.09) \qquad \sum_{i=1}^{n} (\xi_i y_i' - \eta_i x_i') = 0.$$

Suppose

$$(1.1) \qquad 2Q(\xi, \eta) = \sum_{r,s} \frac{\partial^2 F}{\partial x_r \partial x_s} \xi_r \xi_s + 2 \sum_{r,s} \frac{\partial^2 F}{\partial x_r \partial y_s} \xi_r \eta_s + \sum_{r,s} \frac{\partial^2 F}{\partial y_r \partial y_s} \eta_r \eta_s$$

and define

$$(1.11) \qquad Q(\xi\eta, \xi'\eta') = \sum_{r=1}^{n} \xi'_r \frac{\partial Q}{\partial \xi_r} + \sum_{r=1}^{n} \eta'_r \frac{\partial Q}{\partial \eta_r} = Q(\xi'\eta', \xi\eta)$$
$$= \sum_{r,s} \frac{\partial^2 F}{\partial x_r \partial x_s} \xi'_r \xi_s + \sum_{r,s} \frac{\partial^2 F}{\partial x_r \partial y_s} \xi'_r \eta_s + \sum_{r,s} \frac{\partial^2 F}{\partial y_r \partial x_s} \eta'_r \xi_s + \sum_{r,s} \frac{\partial^2 F}{\partial y_r \partial y_s} \eta'_r \eta_s$$

The functions of (1.05) satisfy the equations

$$(1.12) \qquad d\bar{\xi}_i'/dt = a\bar{\xi}_i', \quad d\bar{\xi}_i''/dt = \bar{\xi}_i' + a\bar{\xi}_i'', \quad \cdots, \quad d\bar{\xi}_i^{(\nu)}/dt = \bar{\xi}_i^{(\nu-1)} + a\bar{\xi}_i^{(\nu)},$$
$$d\bar{\eta}_i'/dt = a\bar{\eta}_i', \quad d\bar{\eta}_i''/dt = \bar{\eta}_i' + a\bar{\eta}_i'', \quad \cdots, \quad d\bar{\eta}_i^{(\nu)}/dt = \bar{\eta}_i^{(\nu-1)} + a\bar{\eta}_i^{(\nu)},$$

consequently

$$(1.2) \qquad a\xi_i' = Q_{\eta_i}(\xi', \eta'), \quad \xi_i^{(r-1)} + a\xi_i^{(r)} = Q_{\eta_i}(\xi^{(r)}, \eta^{(r)}),$$
$$a\eta_i' = -Q_{\xi_i}(\xi', \eta'), \quad \eta_i^{(r-1)} + a\eta_i^{(r)} = -Q_{\xi_i}(\xi^{(r)}, \eta^{(r)}),$$
$$r = 2 \cdots \nu;$$

and if $a + a' \neq 0$,

$$Q(\xi^{(r)}\eta^{(r)}, x^{(s)}y^{(s)}) = \sum_{i=1}^{n} [x_i^{(s)} Q_{\xi_i}(\xi^{(r)}, \eta^{(r)}) + y_i^{(s)} Q_{\eta_i}(\xi^{(r)}, \eta^{(s)})]$$
$$= a \sum_{i=1}^{n} (\xi_i^{(r)} y_i^{(s)} - \eta_i^{(r)} x_i^{(s)}) = 0, \quad r = 1,$$
$$(1.3)$$
$$= \sum_{i=1}^{n} (\xi_i^{(r-1)} y_i^{(s)} - \eta_i^{(r-1)} x_i^{(s)})$$
$$+ a \sum_{r=1}^{n} (\xi_i^{(r)} y_i^{(s)} - \eta_i^{(r)} x_i^{(s)}) = 0,$$
$$r = 2 \cdots \nu, \quad s = 1 \cdots \mu.$$

Suppose $a_1, a_2, \cdots, a_M$ the pure imaginary exponents such that $a_r/\sqrt{-1}$

> 0, $(r = 1, \cdots, M)$, while $a_{M+1}, \cdots, a_n$ have their real parts positive; suppose also $a_{n+r} = -a_r$, and let the fundamental set of solutions be so numbered that if the solutions $(\xi_{ki}^{(r)}, \bar{\eta}_{ki}^{(r)})$ $(r = m, m+1, \cdots, m+\nu_k-1)$ form a group corresponding to a_k, the solutions $(\xi^{(r)}{}_{n+k,i}, \bar{\eta}^{(r)}{}_{n+k,i})$ $(r = m, m+1, \cdots, m+\nu_k-1)$ form a group corresponding to a_{n+k}. Under these conditions make the transformation

$$
\begin{aligned}
x_i &= \sum_k (\xi_{ki}^{(\nu k)} u_1^{(k)} + \xi_{ki}^{(\nu k-1)} u_2^{(k)} + \cdots + \xi'_{ki} u_{\nu_k}^{(k)}) \\
&\quad + \sum_k (\xi^{(\nu k)}{}_{n+k,i} v_1^{(k)} + \cdots + \xi'_{n+k,i} v_{\nu_k}^{(k)}), \\
y_i &= \sum_k (\eta_{ki}^{(\nu k)} u_1^{(k)} + \eta_{ki}^{(\nu k-1)} u_2^{(k)} + \cdots + \eta'_{ki} u_{\nu_k}^{(k)}) \\
&\quad + \sum_k (\eta^{(\nu k)}{}_{n+k,i} v_1^{(k)} + \cdots + \eta'_{n+k,i} v_{\nu_k}^{(k)}),
\end{aligned} \tag{1.4}
$$

the first summation extending over all the groups with the exponents $a_1, \cdots, a_n$, the second over the groups with the exponents $a_{n+1}, \cdots, a_{2n}$.

If the transformation (1.4) is made in the right-hand members of (1.02), these become

$$
\begin{aligned}
dx_i/dt &= \sum_k [Q\eta_i(\xi_k^{(\nu k)}, \eta_k^{(\nu k)}) u_1^{(k)} + \cdots + Q\eta_i(\xi_k', \eta_k') u_{\nu_k}^{(k)}] \\
&\quad + \sum_k [Q\eta_i(\xi_{n+k}^{(\nu k)}, \eta_{n+k}^{(\nu k)}) v_1^{(k)} + \cdots + Q\eta_i(\xi'_{n+k}, \eta'_{n+k}) v_{\nu_k}^{(k)}] + F'_i, \\
dy_i/dt &= -\sum_k [Q_{\xi i}(\xi_k^{(\nu k)}, \eta_k^{(\nu k)}) u_1^{(k)} + \cdots + Q_{\xi i}(\xi'_k, \eta'_k) u_{\nu_k}^{(k)}] \\
&\quad - \sum_k [Q_{\xi i}(\xi_{n+k}^{(\nu k)}, \eta_{n+k}^{(\nu k)}) v_1^{(k)} + \cdots + Q_{\xi i}(\xi'_{n+k}, \eta'_{n+k}) v_{\nu_k}^{(k)}] + F_i'', \\
&\qquad\qquad i = 1 \cdots n.
\end{aligned} \tag{1.5}
$$

If for the functions Q_{ξ_i}, Q_{η_i} we substitute their equals from (1.2), and rearrange the linear terms, then for the derivatives $\frac{dx_i}{dt}$, $\frac{dy_i}{dt}$ substitute their equals from (1.4), linear expressions in the variables $u_s^{(r)}$, $v_s^{(r)}$ and their derivatives with the same coefficients are obtained on both sides of the equations, their determinant being the determinant of the fundamental set of solutions and hence different from zero. Solving, we obtain

$$
\begin{aligned}
du_1^{(k)}/dt &= a_k u_1^{(k)} + \phi_1^{(k)}, \qquad dv_1^{(k)}/dt = a_{n+k} v_1^{(k)} + \psi_1^{(k)}, \\
du_2^{(k)}/dt &= u_1^{(k)} + a_k u_2^{(k)} + \phi_2^{(k)}, \qquad dv_2^{(k)}/dt = v_1^{(k)} + a_{n+k} v_2^{(k)} + \psi_2^{(k)}, \\
du_{\nu_k}^{(k)}/dt &= u_{\nu_k-1}^{(k)} + a_k u_{\nu_k}^{(k)} + \phi_{\nu_k}^{(k)}, \qquad dv_{\nu_k}^{(k)}/dt = v_{\nu_k-1}^{(k)} + a_{n+k} v_{\nu_k}^{(k)} + \psi_{\nu_k}^{(k)},
\end{aligned} \tag{1.6}
$$

the functions $\phi_k^{(r)}$, $\psi_k^{(r)}$ being linear combinations of the functions F_i', F_i''.

The function $Q(xy)$, representing the terms of the second order in the

Taylor expansion of F, can easily be calculated from (1.1), (1.11), (1.3), (1.4); we obtain

$$
\begin{aligned}
Q(x,y) &= \tfrac{1}{2}\sum_i [x_i Q_{x_i}(xy) + y_i Q_{y_i}(xy)] \\
&= \sum_{klrs} Q(\xi_{n+k}{}^{\nu_k - r+1}\eta_{n+k}{}^{\nu_k-r+1}, \xi_l{}^{\nu_l-s+1}\eta_l{}^{\nu_l-s+1}) u_s{}^{(l)} v_r{}^{(k)} \\
&= \sum_{i=1}^{M} a_i \epsilon_i u_1{}^{(i)} v_1{}^{(i)} + \sum_{klrs}{}' Q(\xi_{n+k}{}^{\nu_k-r+1}\eta_{n+k}{}^{\nu_k-r+1}, \xi_l{}^{\nu_l-s+1}\eta_l{}^{\nu_l-s+1}) u_s{}^{(l)} v_r{}^{(k)}
\end{aligned} \tag{1.7}
$$

where Σ' denotes a summation over the groups for which the real part of the exponent is different from zero; the coefficient $Q(\xi_{n+k}{}^{(r)}\eta_{n+k}{}^{(r)}\xi_l{}^{(s)}\eta_l{}^{(s)})$ is different from zero only if $a_l + a_{n+k} = 0$.

Since the function F is defined only for real values of its arguments, in general, it will be necessary to give $u_s{}^{(r)}$, $v_s{}^{(r)}$ such values that x_i, y_i, will be real; this result will be obtained if the following conditions are satisfied:

(a) when a_r is a pure imaginary, $a_r = p_r\sqrt{-1}$, $p_r > 0$, $\bar{x}_r$ and $\bar{y}_r$ are real, and $u_1{}^{(r)} = \bar{x}_r + i\bar{y}_r$, $v_1{}^{(r)} = \bar{x}_r - i\bar{y}_r$;

(b) when a_s is real, $a_s = q_s > 0$, $u_k{}^{(s)} = u_{sk}$, $v_k{}^{(s)} = v_{sk}$, where u_{sk} and v_{sk} are real; and (c) when a_r is neither real nor pure imaginary,

$$
\begin{aligned}
&a_r = q_r + iq_r', \quad a_s = q_r - iq_r', \quad q_r > 0, \\
&u_k{}^{(r)} = u_{rk} + iu_{sk} \qquad v_k{}^{(r)} = v_{rk} + iv_{sk} \\
&u_k{}^{(s)} = u_{rk} - iu_{sk} \qquad v_k{}^{(s)} = v_{rk} - iv_{sk}
\end{aligned}
$$

where u_{rk}, v_{rk}, u_{sk}, v_{sk} are real variables.

Of the exponents $a_1, \cdots, a_n$ suppose there are M pure imaginaries, K real, and L pairs of conjugates, each exponent counted just once for each group (1.6). Then these equations take one of the following forms, a_r and a_{L+r} being conjugates:

$$
\begin{aligned}
&\text{(a)} \quad d\bar{x}_i/dt = -p_i\bar{y}_i + X_i, \qquad d\bar{y}_i/dt = p_i\bar{x}_i + Y_i, \quad i = 1 \cdots M. \\
&\text{(1.8)} \quad du_{r1}/dt = q_r u_{r1} + U_{r1}, \qquad dv_{r1}/dt = -q_r v_{r1} + V_{r1}, \\
&\text{(b)} \quad du_{ri}/dt = u_{r,i-1} + q_r u_{ri} + U_{ri}, \quad dv_{ri}/dt = v_{r,i-1} - q_r v_{ri} + V_{ri}, \\
&\qquad\qquad \begin{pmatrix} i = 2 \cdots \nu_r, \\ r = 1 \cdots k \end{pmatrix} \\
&du_{r1}/dt = q_r u_{r1} - q'_r u_{L+r,1} + U_{r1}, \\
&\qquad\qquad dv_{r1}/dt = \quad -q_r v_{r1} + q'_r v_{L+r,1} + V_{r1},
\end{aligned}
$$

$$(1.8)\quad du_{ri}/dt = u_{r,i-1} + q_r u_{ri} - q'_r u_{L+r,i} + U_{ri},$$
$$dv_{ri}/dt = v_{r,i-1} - q_r v_{ri} + q'_r v_{L+r,i} + V_{ri},$$
$$i = 2, \cdots \nu_r;$$

$$(c)\quad du_{L+r,1}/dt = q'_r u_{r1} + q_r u_{L+r,1} + U_{L+r,1},$$
$$dv_{L+r,1}/dt = -q'_r v_{r1} - q_r v_{L+r,1} + V_{L+r,1},$$
$$du_{Lr+,i}/dt = u_{L+r,i-1} + q'_r u_{ri} + q_r u_{L+r,i} + U_{L+r,i},$$
$$dv_{L+r,i}/dt = v_{L+r,i-1} - q_r' v_{r,i} - q_r v_{L+r,i} + V_{L+r,i}.$$
$$i = 2, \cdots \nu_r; \quad r = K+1 \cdots K+L.$$

From the equations (a), (b), (c) can be obtained certain fundamental *differential inequalities* similar to those employed by Bohl;
if for each system of type (b),

$$g_r(u_r) = \sum_{m=1}^{\nu_r} q_r^{2m-1} u_{rm}^2, \quad g_r(v_r) = \sum_{m=1}^{\nu_r} q_r^{2m-1} v_{rm}^2, \quad r = 1 \cdots K,$$

we obtain

$$dg_r(u_r)/dt = q_r^2 u_{r1}^2 + q_r^{2\nu_r} u_{r,\nu_r}^2 + \sum_{m=2}^{\nu_r} (q_r^m u_{rm} + q_r^{m-1} u_{r,m-1})^2$$
$$(1.9) \qquad + 2\sum_{m=1}^{\nu_r} q_r^{2m-1} u_{rm} U_{rm},$$
$$dg_r(v_r)/dt = -q_r^2 v_{r1}^2 - q_r^{2\nu_r} v_{r,\nu_r}^2$$
$$- \sum_{m=2}^{\nu_r} (q_r^m v_{rm} - q_r^{m-1} v_{r,m-1})^2 + 2\sum_{m=1}^{\nu_r} q_r^{2m-1} v_{rm} V_{rm}.$$

For each system of type (c), if

$$g_r(u_r) = \sum_{m=1}^{\nu_r} q_r^{2m-1} u_{rm}^2, \qquad g_r(v_r) = \sum_{m=1}^{\nu_r} q_r^{2m-1} v_{rm}^2,$$

then

$$d/dt[g_r(u_r) + g_r(u_{L+r})] = q_r^2 u_{r1}^2 + q_r^{2\nu_r} u_{r,\nu_r}^2 + \sum_{m=2}^{\nu_r} (q_r^{m-1} u_{r,m-1} + q_r^m u_{rm})^2$$
$$+ q_r^2 u_{L+r,1}^2 + q_r^{2\nu_r} u_{L+r,\nu_r}^2 + \sum_{m=2}^{\nu_r} (q_r^{m-1} u_{L+r,m-1} + q_r^m u_{L+r,m})^2$$
$$(1.91) \qquad + 2\sum_{m=1}^{\nu_r} q_r^{2m-1} (u_{rm} U_{rm} + u_{L+r,m} U_{L+r,m})$$
$$d/dt[g_r(v_r) + g_r(v_{L+r})] = -q_r^2 v_{r1}^2 - q_r^{2\nu_r} v_{r,\nu_r}^2$$
$$- \sum_{m=2}^{\nu_r} (q_r^{m-1} v_{r,m-1} - q_r^m v_{r,m})^2 - q_r^2 v_{L+r,1}^2 - q^{2\nu_r} v_{L+r,\nu_r}^2$$
$$- \sum_{m=2}^{\nu_r} (q_r^{m-1} v_{L+r,m-1} - q_r^m v_{L+r,m})^2 + 2\sum_{m=1}^{\nu_r} q_r^{2m-1} \begin{pmatrix} v_{rm} & V_{rm} \\ + v_{L+r,m} & V_{L+r,m} \end{pmatrix}$$

---

* P. Bohl, "Sur certaines équations différentielles d'un type général utilisables en mécanique," *Bull. Soc. Math. de France*, tome 38 (1910), p. 18.

From the manner in which equations (1.6) were derived, it is seen that if a_k, a_l are conjugate imaginaries, the functions $\phi_i^{(k)}$, $\phi_i^{(l)}$ are conjugates, as are $\psi_i^{(k)}$, $\psi_i^{(l)}$; and the functions X_i, Y_i, U_{rs}, V_{rs} are real functions of the present variables. From (1.08), (1.7), we obtain

$$(1.92) \qquad F = F_0 + \sum_{i=1}^{M} \pm(\bar{x}_i^2 + \bar{y}_i^2) + \sum_{klrs}' A_{r,s}^{k,l} u_{kr} v_{ls} + F_3,$$

in which the constants A_{rs}^{kl} are real.

Suppose (a): $|\bar{x}_i| \leqq a$, $|\bar{y}_i| \leqq a$, $|u_{rs}| \leqq a$, $|v_{rs}| \leqq a$, a region such that the original variables x_i, y_i lie within (D) when the new variables lie within (a). The functions X_i, Y_i, U_{rs}, V_{rs} possess continuous partial derivatives of the first order with respect to the new variables, which vanish with their arguments. If $g(\bar{x}_i, \bar{y}_i, u_{rs}, v_{rs})$ is the largest of the absolute values of these derivatives at the point $\bar{x}_i$, $\bar{y}_i$, u_{rs}, v_{rs}), then

$$|X_i| \leqq g(\tau\bar{x}_i, \tau\bar{y}_i, \tau u_{rs}, \tau y_{rs}) \left[\sum_i (|\bar{x}_i| + |\bar{y}_i|) + \sum_{r,s} (|u_{rs}| + |v_{rs}|)\right], \qquad 0 < \tau < 1,$$

and similar inequalities hold, for other values of τ, for the other functions.

Instead of a single trajectory consider two, represented by the variables

$$\bar{x}_i,\ \bar{y}_i,\ u_{rs},\ v_{rs},\ \ \bar{x}_i + x'_i,\ \ \bar{y}_i + y'_i,\ \ u_{rs} + u'_{rs},\ \ v_{rs} + v'_{rs};$$

since equations (1.8) are valid for both, we obtain a similar set of equations

$$(1.81) \qquad dx_i'/dt = -p_i y_i' + X_i', \quad dy_i'/dt = p_i x_i' + Y_i', \quad \text{etc.},$$

such that if X'_i, Y'_i, U'_{rs}, V'_{rs} represent the differences between the unprimed functions at the two points considered, the new equations (1.81) contain the primed variables and functions where the equations (1.8) contain the unprimed variables and functions. Equations similar to (1.9), (1.91) will also be obtained. We shall have

$$|X_i'| \leqq g\,(\bar{x}_i + \tau x_i', \bar{y}_i + \tau y_i', u_{rs} + \tau u_{rs}', v_{rs} + \tau v_{rs}') \left[\sum_i (|x_i'| + |y_i'|) + \sum_{r,s} (|u_{rs}'| + |v_{rs}'|)\right], \qquad 0 < \tau < 1, \quad \text{etc.}$$

2. *Certain fundamental inequalities.*

Suppose $l_1, l_2, \cdots, l_M, m_1, m_2, \cdots, m_{K+L}$ any set of positive numbers, and $s_1, s_2, \cdots, s_M, t_1, t_2, \cdots, t_M, z_{rs}$ $(s = 1, \cdots, \nu_r,\ r = 1, \cdots, K + L)$ a set of real variables. Then from the inequality

$$\sqrt{m}\,\sqrt{\sum_{i=1}^{m} a_i^2} \geqq \sum_{r=1}^{m} |a_i|$$

and the introduction of the constants

$$\bar{k} \leqq l_i, \quad \bar{k} \leqq m_r q_r^{s-\frac{1}{2}}, \quad \bar{g} \geqq l_i, \quad \bar{g} \geqq m_r q_r^{s-\frac{1}{2}}, \quad N = n - M, \tag{2.1}$$

$$\Gamma = (1/\bar{k}) \sqrt{2M+N}, \qquad \Delta = \bar{g} \sqrt{2M+N}$$

we find

$$\Gamma \sqrt{\sum_{i=1}^{M} l_i^2 (s_i^2 + t_i^2) + \sum_{r=1}^{K} m_r^2 g_r(z_r) + \sum_{r=K+1}^{K+L} m_r^2 [g_r(z_r) + g_r(z_{L+r})]}$$

$$\geqq \sum_{r=1}^{M} (|s_i| + |t_i|) + \sum_{r,s} |z_{rs}| \tag{2.2}$$

$$\Delta \sqrt{\sum_{i=1}^{M} l_i^2 (s_i^2 + t_i^2) + \sum_{r=1}^{K} m_r^2 g_r(z_r) + \sum_{r=K+1}^{K+L} m_r^2 [g_r(z_r) + g_r(z_{L+r})]} \geqq$$

$$\geqq \sum_{i=1}^{M} l_i^2 (|s_i| + |t_i|) + \sum_{r=1}^{K} \sum_{s=1}^{\nu_r} m_r^2 q_r^{2s-1} |z_{rs}|$$

$$+ \sum_{r=K+1}^{K+L} \sum_{s=1}^{\nu_r} m_r^2 q_r^{2s-1} (|z_{rs}| + z_{L+r,s}|)$$

Suppose $q_{K+L+r} = q_{K+r}$; let

$$G(u) = \sum_{r=1}^{K+2L} g_r(u_r)$$

and

$$h_r(u_r) = q_r^2 u_{r1}^2 + q_r^{2\nu_r} u^2_{r,\nu_r} + \sum_{m=2}^{\nu_r} (q_r^m u_{rm} + q_r^{m-1} u_{r,m-1})^2,$$

$$r = 1, 2 \cdots K + 2L \tag{2.3}$$

$$\bar{h}_r(v_r) = q_r^2 v_{r1}^2 + q_r^{2\nu_r} v^2_{r,\nu_r} + \sum_{m=2}^{\nu_r} (q_r^m v_{rm} - q_r^{m-1} v_{r,m-1})^2,$$

$$H(u) = \sum_{r=1}^{K+2L} h_r(u_r), \qquad \bar{H}(v) = \sum_{r=1}^{K+2L} \bar{h}_r(v_r).$$

Since the quadratic forms $h_r(u_r)$, $\bar{h}_r(v_r)$ are positive definite, there exist positive constants λ_r, $\bar{\lambda}_r$ such that these forms satisfy the inequalities

$$h_r(u_r) - \lambda_r g_r(u_r) \geqq 0, \quad \bar{h}_r(v_r) - \bar{\lambda}_r g_r(v_r) \geqq 0,$$

and it will be assumed that these constants have their greatest values.*

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* It follows that the quantities λ_r, $\bar{\lambda}_r$ are roots of the characteristic equations of the pencils of forms

$$h_r(u_r) - \lambda_r g_r(u_r), \quad h_r(v_r) - \bar{\lambda} g_r(v_r),$$

respectively. Since the transformation

$$v_{r1} = u_{r1}, \quad v_{r2} = -u_{r2}, \quad v_{r,2m-1} = u_{r,2m-1}, \quad v_{r,2m} = -u_{r,2m}, \cdots$$

carries the second form into the first, the characteristic equations have the same roots, the smallest of these roots being the values of λ_r, $\bar{\lambda}_r$.

Suppose P, Q constants such that $Q \geqq q_r^{2s-1} \geqq P$, $(s = 1, \cdots \nu_r, r = 1, \cdots K + L)$; then

$$(2.4) \qquad \sum_{r,s} q_r^{2s-1} \mid z_{rs} \mid \leqq \sqrt{NQ}\ \sqrt{G(z)}, \quad \sum_{r,s} \mid z_{rs} \mid \leqq \sqrt{N/P}\ \sqrt{G(z)}.$$

If

$$r' = \sum_{i=1}^{M} l_i^2(x_i'^2 + y_i'^2) + \sum_{r=1}^{K} m_r^2 g_r(v_r') + \sum_{r=K+1}^{K+L} m_r^2[g_r(v_r') + g_{L+r}(v'_{L+r})] - G(u'),$$

then from (1.9), (1.91), if q is the smallest of the constants $\bar{\lambda}_1 \cdots \bar{\lambda}_{K+L}$, $\lambda_1 \cdots \lambda_{K+L}$, and if $r' \leqq 0$, $t = t_0$, we obtain easily

$$dr'/dt \leqq [-q + 2g(\Gamma + \sqrt{N/P})(\Delta + \sqrt{NQ})]G(u'), \quad t = t_0.$$

If in (a),

$$(2.5) \qquad g < \frac{q}{2(\Gamma + \sqrt{N/P})(\Delta + \sqrt{NQ})}$$

then $dr'/dt \leqq 0$.

Suppose this inequality for g satisfied within (a). If the two trajectories lie in (a) for $t_0 \leqq t \leqq t_1$, and $r' \leqq 0$ for $t = t_0$, while $G(u') > 0$, then $dr'/dt < 0$, $t = t_0$. Suppose $r' \geqq 0$ for $t = t_1$; at some time $t' \leqq t_1$, $r' = 0$, and $r' < 0$, $t_0 < t < t'$. Hence there exists also a time $t'' < t'$ for which $dr'/dt = 0$; but at this time $r' < 0$, hence $dr'/dt < 0$. From this contradiction we conclude that $r' < 0$ when $t = t_1$, and $r' < 0$, $t_0 < t \leqq t_1$. Hence from the definition of r', $G(u') > 0$, $t_0 < t \leqq t_1$, and if $G(u') \neq 0$, $t = t_0$, then $G(u') > 0$, $t_0 \leqq t \leqq t_1$.

Under the assumption $r' \leqq 0$, and the preceding inequalities,

$$d/dt\ G(u') = H(u') + 2\left[\sum_{r=1}^{K}\sum_{s} q_r^{2s-1} u_{rs}' U_{rs}' + \sum_{r=K+1}^{K+L}\sum_{s} q_r^{2s-1}(u_{rs}' U_{rs}' + u'_{L+r,s} U'_{L+r,s})\right]$$

$$\geqq H(u') - 2g\left[\sum_{r=1}^{K}\sum_{s} q_r^{2s-1} \mid u_{rs}' \mid + \sum_{r=K+1}^{K+L}\sum_{s} q_r^{2s-1}(\mid u_{rs}' \mid + \mid u'_{L+r,s} \mid)\right]$$

$$[\sum_{i}(\mid x_i' \mid + \mid y_i' \mid + \sum_{r,s} \mid u_{rs}' \mid + \sum_{r,s} \mid v'_{rs} \mid]$$

$$\geqq H(u') - 2g\sqrt{NQ}\sqrt{G(u')}(\Gamma + \sqrt{N/P})\sqrt{G(u')}$$

$$\geqq [q - 2g(\Gamma + \sqrt{N/P})\sqrt{NQ}]\, G(u').$$

If the constant g satisfies (2.5), there can be found a constant σ such that

$$q > \sigma > 2g\sqrt{NQ}(\Gamma + \sqrt{N/P});$$

and

$$dG(u')/dt > (q-\sigma)G(u'),$$

whence

$$[G(u')]_{t=t_1} > e^{(q-\sigma)(t_1-t_0)}[G(u')]_{t=t_0}.$$

From the preceding results we have the theorem: *If* $\sigma < q$, and $l_1, l_2, \cdots, l_M, m_1, m_2, \cdots, m_{K+L}$ *are any positive numbers, then if a neighborhood* (a) *of the origin is so chosen that within it the partial derivatives of* X_i, Y_i, U_{rs}, V_{rs} *are less in absolute value than either of the quantities*

$$\frac{q}{2(\Gamma+\sqrt{N/P})(\Delta+\sqrt{NQ})}, \quad \frac{\sigma}{2\sqrt{NQ}(\Gamma+\sqrt{N/P})}$$

where Δ, Γ, P, Q, *are the constants previously defined, this neighborhood has the following properties:* 1. *if the coordinates of two trajectories lie in* (a) *for* $t_0 \leqq t \leqq t_1$ *and if when* $t = t_0$, $G(u') \neq 0$,

$$r' = \sum_{i=1}^{M} l_i^2 (x_i'^2 + y_i'^2) + \sum_{r=1}^{K} m_r^2 g_r(v_r') + \sum_{r=K+1}^{K+L} m_r^2 [g_r(v_r') + g_r(v'_{L+r})] - G(u') \leqq 0,$$

then $r' < 0$ *for* $t_0 < t \leqq t_1$, *and*

$$[G(u')]_{t=t_1} > e^{(q-\sigma)(t_1-t_0)}[G(u')]_{t=t_0}.$$

2. *If the coordinates defining two trajectroies lie in* (a) *for* $t_0 \geqq t \geqq t_1$ $(t_0 > t_1)$, *and if for* $t = t_0$, $G(v') \neq 0$, and

$$s' = \sum_{i=1}^{M} l_i^2 (x_i'^2 + y_i'^2) + \sum_{r=1}^{K} m_r^2 g_r(u'_r) + \sum_{r=K+1}^{K+L} m_r^2 [g_r(u_r') + g_r(u'_{L+r})] - G(v') \leqq 0,$$

then $s' < 0$ *for* $t_0 > t \geqq t_1$, *and*

$$[G(v')]_{t=t_1} > e^{(q-\sigma)(t_0-t_1)}[G(v')]_{t=t_0}.$$

The last part of the theorem can be demonstrated in the same manner.

3. *An application of the energy integral.*

In what follows it will be assumed that the form $\sum_{i=1}^{M} \pm (\bar{x}_i^2 + \bar{y}_i^2)$ is definite; suppose the constant A so large that for all values of the variables concerned,

$$-A[G(u) + G(v)] \leqq \sum_{klrs}' A_{rs}^{k,l} u_{kr} v_{ls} \leqq A[G(u) + G(v)].$$

Assume μ, ν, ρ positive constants which satisfy the inequality

$$\rho + \mu + \rho\mu + 4\nu(A + \mu/P) < 1.$$

Since the first and second partial derivatives of F_3 vanish with its arguments, we can assume that in (E), a sub-region of (a),

$$|F_3| \leqq \mu\,[\sum_i (\bar{x}_i^2 + \bar{y}_i^2) + \sum_{r,s} (u_{rs}^2 + v_{rs}^2)]. \tag{3.1}$$

Then suppose $\delta > 0$ such that the region defined by

$$\sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) \leqq \delta^2, \tag{R'}$$

$$G(u) \leqq \nu\delta^2, \qquad G(v) \leqq \nu\delta^2 \tag{S}$$

lies within (E), and let (R″) be the region

$$\sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) \leqq \rho\delta^2. \tag{R″}$$

From the energy integral, if a trajectory lies in (R′), (S) at $t = t_1$, in (R″), (S) at $t = t_2$,

$$\begin{aligned}
[\sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2)]_{t=t_1} &= \pm\,[\pm \sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) + \sum_{klrs}{}' A_{rs}{}^{kl} u_{kr} v_{ls} + F_3]_{t=t_2} \\
&\quad \mp\,[\sum_{klrs}{}' A_{rs}{}^{kl} u_{kr} v_{ls} + F_3]_{t=t_1} \\
&\leqq \rho\delta^2 + 4A\nu\delta^2 + \mu[\delta^2 + \rho\delta^2 + 1/P(2\nu\delta^2 + 2\nu\delta^2)] \\
&\leqq [\rho + \mu + \rho\mu + 4\nu(A + \mu/P)]\delta^2 < \delta^2.
\end{aligned} \tag{3.2}$$

Let (H) be a region lying within (E), and such that in (H) the partial derivatives of X_i, Y_i, U_{rs}, V_{rs} are less in absolute value than θ where

$$\theta < \frac{q\nu}{2\sqrt{NQ\nu}\,(\sqrt{2M} + 2\sqrt{N\nu/P})}.$$

Suppose d so small that the region

$$\sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) \leqq d^2, \quad G(u) \leqq \nu d^2, \quad G(v) \leqq \nu d^2$$

lies within (H), and construct the regions

$$\sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) \leqq d^2, \tag{A}$$

$$\sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) \leqq \rho d^2, \tag{a}$$

$$G(u) \leqq \nu d^2, \quad G(v) \leqq \nu d^2. \tag{B}$$

It results that *if* $(\bar{x}_i, \bar{y}_i, u_{rs}, v_{rs})$ *lies in* (a, B) *for* $t = t_2$, *in* (A, B) *for* $t = t_1$, *and if* $G(u) = vd^2$ *for* $t = t_1$, *then* $dG(u)/dt > 0$, $t = t_1$.

For when $t = t_1$, from (3.2)

$$dG(u)/dt \geqq H(u) - 2\theta\sqrt{NQG(u)}$$
$$[\sqrt{2M}\,\sqrt{\sum_i (\bar{x}_i^2 + \bar{y}_i^2)} + \sqrt{2N/P}\,\sqrt{G(u) + G(v)}]$$
$$\geqq [qv - 2\theta\sqrt{NQv}\,(\sqrt{2M} + 2\sqrt{Nv/P})]d^2 > 0.$$

4. *Existence theorems.*

With the definitions of the domains (A), (a), (B) of the preceding paragraph, we obtain at once, by the reasoning of the first paper (§ 4, pp. 40-42) the following theorems: 1. *To every set of values* $(\bar{x}_i, \bar{y}_i)$, (v_{rs}) *of the region*

$$\sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) \leqq \rho d^2, \quad G(v) \leqq vd^2,$$

can be adjoined a set of values (u_{rs}) *of the region* $G(u) \leqq vd^2$ *such that the trajectory having these initial values at* $t = t_0$ *remains in the region* (A, B) *for* $t > t_0$.

2. *To every set of values* $(\bar{x}_i, \bar{y}_i)$, (u_{rs}) *of the region*

$$\sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) \leqq \rho d^2, \quad G(u) \leqq vd^2,$$

can be adjoined a set of values (v_{rs}) *of the region* $G(v) \leqq vd^2$ *such that the trajectory having these initial values at* $t = t_0$ *lies in the region* (A, B) *for* $t < t_0$.

Let (H′) be the region (H) for the particular set of values $l_1 = l_2 = \cdots = l_M = 1$, $m_1 = m_2 = \cdots = m_{K+L} = m < 1$; from § 2 the quantity r' formed for two trajectories which lie in (H′) for $t \geq t_0$,

$$r' = \sum_{i=1}^{M} (x_i'^2 + y_i'^2) + m^2 G(v') - G(u')$$

must be positive or zero, otherwise $G(u')$ would increase indefinitely with t. Consequently

$$G(u') \leqq \sum_{i=1}^{M} (x_i'^2 + y_i'^2) + m^2 G(v'), \quad t \geqq t_0; \tag{4.1}$$

and *if two trajectories lie in* (H′) *for* $t \geqq t_0$, *and if at any instant* $t' > t_0$, $x_i' = y_i' = v_{rs}' = 0$, *then* $u_{rs}' = 0$, *and the two trajectories coincide.*

In the same manner we obtain the result that *if two trajectories lie in* (H') *for* $t \leqq t_0$, *and if at any instant* $t' < t_0$, $x_i' = y_i' = u_{rs}' = 0$, *then* $v_{rs}' = 0$, *and the two trajectories coincide.*

From the last two theorems it follows that the set of values (u_{rs}) corresponding to the given set of values $(\bar{x}_i, \bar{y}_i, v_{rs})$ in the first existence theorem above is uniquely determined; these values, functions of the remaining variables, will be denoted by $[u_{rs}]$: similarly, the values (v_{rs}) whose existence is asserted in the second existence theorem, will be denoted by $[v_{rs}]$. The continuity of these functions follows at once from (4.1) and the corresponding inequality

$$G(v') \leqq \sum_{i=1}^{M} (x_i'^2 + y_i'^2) + m^2 G(u'), \quad t \leqq t_0. \tag{4.2}$$

If a set of values $(\bar{x}_i, \bar{y}_i)$ satisfying (a), § 3, is fixed, the functions

$$u_{rs} - [u_{rs}], \quad v_{rs} - [v_{rs}], \quad s = 1 \cdots \nu_r, \quad r = 1 \cdots K + 2L$$

are continuous in u_{rs}, v_{rs} within and on the boundary of

$$G(u) \leqq \nu d^2, \qquad G(v) \leqq \nu d^2. \tag{B}$$

If no set of values of u_{rs}, v_{rs} exists for which these functions vanish simultaneously, from a theorem of Bohl a set of values must exist on the boundary for which one of the systems of equations must hold:

$$u_{rs} - [u_{rs}] = -\lambda' u_{rs}, \quad v_{rs} - [v_{rs}] = -\lambda' v_{rs}, \quad \lambda' > 0,$$

$$(1 + \lambda') u_{rs} = [u_{rs}], \quad (1 + \lambda') v_{rs} = [v_{rs}], \quad s = 1 \cdots \nu_r, \quad r = 1 \cdots K + 2L.$$

Consequently,

$$(1 + \lambda')^2 G(u) = G([u]) \text{ or } (1 + \lambda')^2 G(v) = G([v])$$

and therefore either

$$(1 + \lambda')^2 \nu d^2 = G([u]) \text{ or } (1 + \lambda')^2 \nu d^2 = G([v]).$$

But this is impossible, since the values $[u_{rs}]$, $[v_{rs}]$ satisfy the inequalities (B). Consequently values must exist for which

$$u_{rs} = [u_{rs}], \quad v_{rs} = [v_{rs}]$$

from which we obtain the theorem: *to every set of values of* (a) *can be adjoined a set of values of* (B) *such that the trajectory possessing these initial values at any instant remains in the region* (A, B) *for all values of* t.

If two trajectories remain in (A, B) for all values of t, we shall have

$$\sum_{i=1}^{M}(x_i'^2+y_i'^2)+m^2G(v')\geqq G(u'),$$

$$\sum_{i=1}^{M}(x_i'^2+y_i'^2)+m^2G(u')\geqq G(v'),$$

from which

$$\sum_{i=1}^{M}(x_i'^2+y_i'^2)\geqq\frac{1-m^2}{2}\ [G(u')+G(v')].$$

Hence if at any instant $x_i'=y_i'=0$, then $u_{rs}'=v_{rs}'=0$, and the two trajectories coincide.

5. *An inequality.*

Let l, m, h be positive, and let α, β satisfy the inequalities $\alpha\leqq l$, $mq_r^{s-\frac{1}{2}}$, $q_r^{s-\frac{1}{2}}\leqq\beta$. Assume k a number satisfying

$$k<\frac{q\alpha}{8\beta(M+N)},\quad k<\frac{h\alpha}{2\beta(M+N)}\ .$$

If

$$\omega=e^{2ht}[l^2\sum_{i=1}^{M}(x_i'^2+y_i'^2)+m^2G(u')-G(v')],$$

where x_i', y_i' u_{rs}', v_{rs}' are the differences for two trajectories in the region (C) where the partial derivatives of X_i', Y_i', U_{rs}', V_{rs}' are less in absolute value than k, then from (1.81),

$$d\omega/dt=2h\omega+e^{2ht}[m^2H(u')+2l^2\sum_{i=1}^{M}(x_i'X_i'+y_i'Y_i')+\sum_{r,s}2m^2q_r^{2s-1}u_{rs}'U_{rs}']$$

$$-e^{2ht}[-\bar{H}(v')+2\sum_{r,s}q_r^{2s-1}v_{rs}'V_{rs}']$$

Now

$$|X_i'|,\ |Y_i'|,\ |U_{rs}'|,\ |V_{rs}'|\leqq k[\sum_i(|x_i'|+|y_i'|)+\sum_{r,s}(|u_{rs}'|+|v_{rs}'|)]$$

$$\leqq k/\alpha\sqrt{2(M+N)}\ \sqrt{\sum_i l^2(x_i'^2+y_i'^2)+m^2G(u')+G(v')}$$

Also,

$$l^2\sum_i(|x_i'|+|y_i'|)+m^2\sum_{r,s}q_r^{2s-1}|u_{rs}'|+\sum_{r,s}q_r^{2s-1}|v_{rs}'|$$

$$\leqq\beta\ [l\sum_i(|x_i'|+|y_i'|)+m\sum_{r,s}q_r^{s-\frac{1}{2}}|u_{rs}'|+\sum_{r,s}q_r^{s-\frac{1}{2}}|v_{rs}'|]$$

$$\leqq\beta\sqrt{2(M+N)}\ \sqrt{l^2\sum_i(x_i'^2+y_i'^2)+m^2\ G(u')+G(v')}$$

and consequently

$$d\omega/dt \geqq 2h\omega + e^{2ht}[qm^2G(u') + qG(v') - 2\ \frac{k\beta}{a}\ (2M+2N)$$
$$\{\sum_{i=1}^{M} l^2(x_i'^2 + y_i'^2) + m^2G(u') + G(v')\}]$$
$$\geqq 2[h - \frac{k\beta}{a}\ (2M+2N)]\omega$$
$$+ e^{2ht}[q - \frac{8\beta k}{a}\ (M+N)]G(v') + qm^2G(u')e^{2ht}$$
$$\geqq 2\rho\omega + \theta e^{2ht}G(v'), \quad (\rho = h - \frac{2k\beta}{a}\ (M+N), \quad \theta = q - \frac{8\beta k}{a}\ (M+N)).$$

From which

$$d/dt\ [e^{-2\rho t}\omega] \geqq 0.$$

If ω is positive for any value of $t > t_0$, ω will therefore increase until one of the trajectories leaves the region (C); if $\omega = 0$ for $t = t_0$ but the two trajectories do not coincide, $G(v') > 0$, hence $d\omega/dt > 0$ and ω takes on positive values. Hence if both trajectories remain in (C) for $t > t_0$, and ω is bounded, then $\omega < 0$ unless the two trajectories coincide.

6. *Asymptotic character of certain trajectories.*

Suppose h a positive constant less than q, q_r, and suppose A, B so small that the region

(J) $$|\bar{x}_i|, |\bar{y}_i| \leqq A, \qquad |u_{rs}|, |v_{rs}| \leqq B$$

lies within (a) and in (J) the partial derivatives of X_i, Y_i, U_{rs}, V_{rs} are less in absolute value than η, a constant to be fixed later. Then we have the theorem: *The constants η, $\bar{H}$, H can be so chosen that if $H < A$, $H < B$, and (K) is the region*

(K) $$|\bar{x}_i|, |\bar{y}_i| \leqq A - H, \qquad |u_{rs}|, |v_{rs}| \leqq B - H,$$

then if a trajectory σ lies in (K) *for $t \geqq t_0$, suppose (v_{rs}^0) its set of v-values at $t = t_0$: if an arbitrary set of v-values is chosen such that*

$$\sum_{r,s} (v_{rs} - v_{rs}^0)^2 (q_r - h)^{2s-1} \leqq \bar{H}^2,$$

then there is one and only one trajectory whose v-values coincide with these for $t = t_0$, which remains in (J) *for $t > t_0$, and approaches σ asymptotically as $t \to \infty$, in such a manner that the differences of the coordinates, multiplied by e^{ht}, remain bounded for $t > t_0$.*

Let the coordinates of σ be represented by the variables $\bar{\bar{x}}_i$, $\bar{\bar{y}}_i$, $\bar{u}_{rs}$, $\bar{v}_{rs}$, and suppose ϵ so small that the region

$$|\bar{x}_i|, |\bar{y}_i| \leqq A + 2(e^{h\epsilon} - 1)H, \quad |u_{rs}|, |v_{rs}| \leqq B + 2(e^{h\epsilon} - 1)H,$$

lies in (a). Choose a constant $\theta < \epsilon$ so small that for $t_0 - \theta \leqq t \leqq t_0$,

$$|\bar{\bar{x}}_i|, |\bar{\bar{y}}| \leqq A - H + (e^{h\epsilon} - 1)H, \quad |\bar{u}_{rs}|, |\bar{v}_{rs}| \leqq B - H + (e^{h\epsilon} - 1)H.$$

Then if

$$|\xi_i|, |\eta_i| \leqq e^{h(\epsilon-\theta)}H, \quad |z_{rs}|, |w_{rs}| \leqq e^{h(\epsilon-\theta)}H, \tag{6.01}$$

we shall have

$$|\bar{\bar{x}}_i + e^{-h\tau}\xi_i|, \quad |\bar{\bar{y}}_i + e^{-h\tau}\eta_i| \leqq A + 2(e^{h\epsilon} - 1)H,$$
$$|\bar{u}_{rs} + e^{-h\tau}z_{rs},| \quad |\bar{v}_{rs} + e^{-h\tau}w_{rs}| \leqq B + 2(e^{h\epsilon} - 1)H.$$
$$\tau = t - t_0 > -\theta,$$

Making the substitution

$$\bar{x}_i = \bar{\bar{x}}_i + e^{-h\tau}\xi_i, \quad \bar{y}_i = \bar{\bar{y}}_i + e^{-h\tau}\eta_i, \tag{6.1}$$
$$u_{rs} = \bar{u}_{rs} + e^{-h\tau}z_{rs}, \quad v_{rs} = \bar{v}_{rs} + e^{-h\tau}w_{rs}$$

we obtain from (1.8) the equations for the new variables ξ_i, η_i, z_{rs}, w_{rs}. These become

(a) $d\xi_i/dt = h\xi_i - p_i\eta_i + \bar{X}_i', \quad d\eta_i/dt = p_i\xi_i + h\eta_i + \bar{Y}_i', \quad i = \cdots M.$

(b) $dz_{r1}/dt = (h + q_r)z_{r1} + \bar{U}_{r1}', \quad dw_{r1}/dt = (h - q_r)w_{r1} + \bar{V}_{r1}'$
$$i = 2 \cdots \nu_r,$$
$$dz_{ri}/dt = z_{r,i-1} + (h + q_r)z_{ri} + \bar{U}_{ri}',$$
$$dw_{ri}/dt = w_{r,i-1} + (h - q_r)w_{ri} + \bar{V}_{ri}', \qquad r = 1 \cdots K.$$

$$dz_{r1}/dt = (h + q_r)z_{r1} + \bar{U}_{r1}', \quad dw_{r1}/dt = (h - q_r)w_{r1} + \bar{V}_{r1}', \tag{6.2}$$
$$dz_{r,i}/dt = z_{r,i-1} + (h + q_r)z_{ri} - q_r'z_{L+r,i} + \bar{U}_{ri}',$$
$$dw_{ri}/dt = w_{r,i-1} + (h - q_r)w_{ri} + q_r'w_{L+r,i} + \bar{V}_{ri}',$$

(c) $dz_{L+r,1}/dt = q_r'z_{r1} + (h + q_r)z_{L+r,1} + \bar{U}'_{L+r,1},$
$$dw_{L+r,1}/dt = -q_r'w_{r1} + (h - q_r)w_{L+r,1} + \bar{V}'_{L+r,1},$$
$$dz_{L+r,i}/dt = z_{L+r,i-1} + q_r'z_{ri} + (h + q_r)z_{L+r,i} + \bar{U}'_{L+r,i},$$
$$dw_{L+r,i}/dt = w_{L+r,i-1} - q_r'w_{ri} + (h - q_r)w_{L+r,i} + \bar{V}'_{L+r,i},$$
$$i = 2 \cdots \nu_r, \quad r = K + 1 \cdots K + L.$$

where

$$\bar{X}_i' = e^{h\tau}[X_i(\bar{x}_i, \bar{y}_i, u_{rs}, v_{rs}) - X_i(\bar{\bar{x}}_i, \bar{\bar{y}}_i, \bar{u}_{rs}, \bar{v}_{rs})],$$
$$\bar{U}_{rs}' = e^{h\tau}[U_{rs}(\bar{x}_i, \bar{y}_i, u_{rs}, v_{rs}) - U_{rs}(\bar{\bar{x}}_i, \bar{\bar{y}}_i, \bar{u}_{rs}, \bar{v}_{rs})], \text{ etc.}$$

the variables $\bar{x}_i$, $\bar{y}_i$, u_{rs}, v_{rs} being given by the substitution (6.1).

The right-hand members of (6.2) are bounded and satisfy a Lipschitz condition for $t \geqq t_0 - \theta$, if the dependent variables are in the region specified above, hence the existence of solutions in a certain neighborhood of $t = t_0$ is assured.

Suppose the forms $G_h(z)$, $\bar{G}_h(w)$ defined by

$$G_h(z) = \sum_{r,s} (q_r + h)^{2s-1} z_{rs}^2, \quad \bar{G}_h(w) = \sum_{r,s} (q_r - h)^{2s-1} w_{rs}^2.$$

If $t \geqq t_0$, and the inequalities

$$\sum_i (\xi_i^2 + \eta_i^2) + G_h(z) \leqq H^2, \quad G_h(w) \leqq \bar{H}^2, \tag{6.3}$$

are satisfied, then if $H_h(z)$ is obtained from $H(z)$ by replacing q_r by $q_r + h$, while $\bar{H}_h(w)$ is obtained from $\bar{H}(w)$ by replacing q_r by $q_r - h$, we obtain as in § 2,

$$\frac{d}{dt}\left[\sum_{i=1}^{M} (\xi_i^2 + \eta_i^2) + G_h(z)\right] \geqq 2h \sum_{i=1}^{M} (\xi_i^2 + \eta_i^2) + H_h(z)$$

$$- 2\left[\sum_{i=1}^{M} (|\xi_i| \cdot |\bar{X}_i'| + |\eta_i| \cdot |\bar{Y}_i'|) + \sum_{r,s} (q_r + h)^{2s-1} |z_{rs}| \cdot |\bar{U}_{rs}'|\right]$$

$$\geqq 2h \sum_{i=1}^{M} (\xi_i^2 + \eta_i^2) + H_h(z)$$

$$- 2\eta \left[\sum_{i=1}^{M} (|\xi_i| + |\eta_i|) + \sum_{r,s} (q_r + h)^{2s-1} |z_{rs}|\right]$$

$$\left[\sum_{i=1}^{M} (|\xi_i| + |\eta_i|) + \sum_{r,s} (|z_{rs}| + |w_{rs}|)\right]$$

If $\bar{\mu}$, $\bar{\nu}$, P_h, Q_h, $\bar{P}_h$, $\bar{Q}_h$ are constants such that

$$H_h(z) \geqq \bar{\mu} G_h(z), \quad \bar{H}_h(w) \geqq \bar{\nu} \bar{G}_h(w),$$

$$Q_h \geqq (q_r + h)^{2s-1} \geqq P_h, \quad \bar{Q}_h \geqq (q_r - h)^{2s-1} \geqq \bar{P}_h,$$

then

$$\frac{d}{dt}\left[\sum_{i=1}^{M} (\xi_i^2 + \eta_i^2) + G_h(x)\right] \geqq 2h \sum_{i=1}^{M} (\xi_i^2 + \eta_i^2) + \bar{\mu} G_h(z)$$

$$- 2\eta \left[\sqrt{2M \sum (\xi_i^2 + \eta_i^2)} + \sqrt{N Q_h}\, \sqrt{G_h(z)}\right] \times$$

$$\times \left[\sqrt{2M \sum_i (\xi_i^2 + \eta_i^2)} + \sqrt{\frac{N}{P_h} G_h(z)} + \sqrt{\frac{N}{\bar{P}_h} \bar{G}_h(w)}\right]$$

If $2\bar{\rho}$ is the smaller of $2h$, $\bar{\mu}$ and if $\bar{H} < H/\sqrt{P_h}$, $\bar{H} < H/\sqrt{\bar{P}_h}$, $\bar{H} < H$ while

$$\sum_i (\xi_i^2 + \eta_i^2) + G_h(z) = \bar{H}^2, \quad \bar{G}_h(w) \leqq H^2$$

then

$$(6.4)\quad d/dt\,[\sum_i (\xi_i^2+\eta_i^2)+G_h(z)] \leqq 2\{\bar\rho-\eta[\sqrt{2M}+\sqrt{N/P_h}+\sqrt{N/\bar P_h}](\sqrt{2M}+\sqrt{NQ_h})\}\bar H^2.$$

We shall assume η so small that the expression on the right multiplying $\bar H^2$ is positive; then the left-hand member is also positive.

Also,

$$\frac{d}{dt}\bar G_h(w) \leqq -\bar H(w)+2\eta\sum_{r,s}(q_r-h)^{2s-1}|\,w_{rs}\,|\,[\Sigma(|\,\xi_i\,|+|\,\eta_i\,|)+\sum_{r,s}(|\,z_{rs}\,|+|\,w_{rs}\,|)]$$

$$\leqq -\bar H(w)+2\eta\sqrt{Q_hN}\left[\sqrt{2M\sum_i(\xi_i^2+\eta_i^2)}+\sqrt{\frac{N}{P_h}\,G_h(z)}+\sqrt{\frac{N}{\bar P_h}\,\bar G_h(w)}\,\right]\times\sqrt{\bar G_h(w)}$$

Now suppose $\bar G_h(w)=\bar H^2$; then if $\sum_i(\xi_i^2+\eta_i^2)+G_h(z)\leqq\bar H^2$,

$$(6.4)\quad \frac{d}{dt}\,\bar G_h(w)\leqq[-\bar\nu+2\eta\sqrt{Q_hN}\,(\sqrt{2M}+\sqrt{N/P_h}+\sqrt{N/\bar P_h})]\bar H^2.$$

it will be assumed that η is so small that the term on the right multiplying H^2 is negative. From the choice of $\bar H$ it follows that when the inequalities (6.3) are satisfied, the first inequalities for the present variables are satisfied also. It will be assumed that A, B are so small that (J) belongs to the region (C) of § 5.

On the basis of these inequalities, we can show * that to every set of w-values satisfying $\bar G_h(w)\leqq\bar H^2$ at least one set of values $(\xi_i,\ \eta_i,\ z_{rs})_0$ can be chosen satisfying

$$(6.5)\qquad \sum_i(\xi_i^2+\eta_i^2)+G_s(z)\leqq\bar H^2$$

such that $(w_{rs})_0$, $(\xi_i,\ \eta_i,\ z_{rs})_0$ are the values for $t=t_0$ of a solution remaining in the region (6.5) for $t>t_0$.

Now suppose a set of v-values (V_{rs}) chosen, such that $\bar G_h(V-v^0)\leqq\bar H^2$, and let a solution of equations (6.2) be determined in accordance with the existence theorem above, such that at $t=t_0$, $w_{rs}=-v_{rs}{}^0+V_{rs}$, $(s=1$

* Bohl, *l. c.*, p. 230. The proof of the proposition stated above is so similar that it is unnecessary to repeat it here.

$\cdots \nu_r$, $r = 1 \cdots K + 2L$), while the solution remains in the region designated for $t > t_0$; for $t > t_0$ the variables ξ_i, η_i, z_{rs}, w_{rs} satisfy

$$|\bar{\bar{x}}_i + e^{-h\tau}\xi_i|, \quad |\bar{\bar{y}}_i + e^{-h\tau}\eta_i| \leqq A,$$
$$|\bar{u}_{rs} + e^{-h\tau}z_{rs}|, \quad |\bar{v}_{rs} + e^{-h\tau}w_{rs}| \leqq B$$

while for $t = t_0$, ${v_{rs}}^0 + w_{rs} = V_{rs}$. The trajectory defined by

$$\bar{x}_i = \bar{\bar{x}}_i + e^{-h\tau}\xi_i, \quad \bar{y}_i = \bar{\bar{y}}_i + e^{-h\tau}\eta_i, \quad i = 1 \cdots M,$$
$$u_{rs} = \bar{u}_{rs} + e^{-h\tau}\xi_{rs}, \quad v_{rs} = \bar{v}_{rs} + e^{-h\tau}w_{rs},$$
$$s = 1 \cdots \nu_r, \quad r = 1 \cdots K + 2L$$

satisfies the first part of the theorem; it is evident that the differences $(e^{-h\tau}\xi_i, e^{-h\tau}\eta_i, e^{-h\tau}z_{rs}, e^{-h\tau}w_{rs})$ multiplied by e^{ht}, remain bounded.

The second part of the theorem follows easily from § 5. In these inequalities suppose $l = m = 1$, and suppose the quantity ω constructed for two solutions remaining in (J) for $t \geqq t_0$, in such a manner that the coordinate differences, multiplied by e^{ht}, remain bounded for $t > t_0$, the constant h being such as to satisfy the inequalities of the present paragraph. Then

$$\omega = e^{2ht}\left[\sum_{i=1}^{M}(x_i'^2 + y_i'^2) + G(u') - G(v')\right]$$

is bounded, and consequently $\omega \leqq 0$ for $t > t_0$. If the differences v_{rs}' vanish at $t = t_0$, it follows that the remaining differences x_i', y_i', u_{rs}', vanish also, and the two trajectories coincide. Hence when the v-values are fixed, only one trajectory having the properties stated can exist.

Applications of the Determinant and Permanent Tensors to Determinants of General class and Allied Tensor Functions.

By Clyde M. Cramlet.

1. *Introduction.* The geometric scheme of representing the elements of a detetrminant as an arrangement of rows and columns or of layers, sections and sheets * becomes cumbersome in the general study of determinants of elements with several indices. The natural analytical method in these cases is to confine attention to the general term of the determinant. Various methods are used to determine the signs of this term and to exclude the terms that should not appear. The method most like that used here is the scheme of Cayley † of using a square determinant in this capacity. Recently ‡¶ papers have appeared dealing with square determinants by a new method. It is along these lines that the work of this paper will proceed. Tensor notation will be used throughout but that is merely for considerations of symmetry. The results are valid for any tensor character that might be assigned. It is important to note, however, that with the transformation character that we shall assign our elements, the laws of transformation of the functions of the elements can be read from the indices. The modifications for different transformations on the elements will be at once apparent. To illustrate, suppose g_{rst} is one of n^3 elements that appears in our work, and that it is a coefficient in the trilinear form $g_{\rho\sigma\tau}x^\rho y^\sigma z^\tau$, then since the ground form is invariant the indices in g_{rst} are all covariant. The transformation character of the determinant or whatever function of these elements that is discussed in our work can be read off at once. On the other hand suppose the g_{rst} are the n^3 coefficients in a system of n quadratics $g_{r\sigma\tau}x^\sigma x^\tau$, $r = 1 \cdots n$, then the first index on the g is not of tensor (or transformation) character. It might be well in this case to indicate it as $g_{\bar{r}st}$. The modifications for this situation can be made in any equations in which the first index is not multiplied by inner multiplication by a tensor.

* Hedrick, *Annals of Mathematics*, Vol. 1 (1899), page 49.

† *Trans. of Cambridge Phil. Soc.*, London, Vol. 8, Part 1, 1844 (2-43), S. 11, p. 85-8. (Reference from Lecat's *Leçons sur la Théorie des Déterminants à n dimensions.*)

‡ Murnaghan, *Math. Monthly*, XXXII, 1925, p. 233.

¶ The writer: *Annals of Mathematics*, Vol. 27, No. 4, September, 1926. This will be referred to hereafter as paper I.

2. *Definitions and Conventions.* We define a tensor * by the equation,

$$\gamma^{a_1 \cdots a_n}_{b_1 \cdots b_n} = \sum_p A^{(p)} \delta^{a_1}_{b_1} \cdots \delta^{a_n}_{b_n} \tag{2.1}$$

where the $n!$ terms formed by permuting the a's and multiplying each by an arbitrary scalar, are added. The $n!$ scalors are distinguished by the $n!$ permutations p. When the scalors are ± 1 according as the permutation p is even or odd we have

$$\delta^{a_1 \cdots a_n}_{b_1 \cdots b_n} \tag{2.2}$$

This tensor vanishes unless the a's and b's are the same set of distinct integers when it equals ± 1 according as the substitution to bring them to the same order is even or odd. This will be called the *determinant tensor.*† When the scalars are all equal to 1 we have

$$\pi^{a_1 \cdots a_n}_{b_1 \cdots b_n} \tag{2.3}$$

which will be called the *permanent tensor.* It vanishes unless the a's and b's are identical sets. For the non-vanishing cases the values are 1 if the sets are composed of distinct integers. If there are q groups with $p_1 \cdots p_q$ repeated or equal indices $p_1 + \cdots p_q \geqslant n$ the value of the tensors is $p_1! + \cdots p_q!$.

An inner product of γ and a set of elements such as

$$\gamma^{\rho_1 \cdots \rho_n}_{s_1 \cdots s_n} \quad g_{r_1 \rho_1} \cdots g_{r_n \rho_n} \tag{2.4}$$

will be called a *tensor function* of the elements. The γ's are of the tensor character indicated yet they are absolute invariants. If a δ is used in place of the γ we have a *determinant function,* and likewise, using a π, a *permanent function.* For functions involving elements with more than two indices we may use combinations of these three. The *degree* of a tensor function will be equal to the number of elements occurring in a term. This term will replace the less significant word, *order.* The set of n indices in the same position on the elements of a term will be called a *range.* Greek letters, only, when repeated will indicate a summation. Repeated letters are called umbral. For the non-repeated we venture the name *virtual.* The virtual ranges on the elements will be called fixed as is customary. The *class* is one greater than

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* Properties of this tensor were given in a paper read before the Society April 3, 1926.

† Murnaghan has suggested the name "generalized Kronecker symbol." Either (2.2) or (2.3) for $n = 1$ becomes the Kronecker symbol.

the number of umbral ranges, i. e. one greater than the number of tensor multipliers when each multiplier has one virtual range. If there is but one fixed range the class is equal to the number of ranges.

These definitions and certain general properties will be illustrated by the function,

$$g_{[r_n](s_n)(t_n)} = \delta^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n} \; \delta^{\tau_1 \cdots \tau_n}_{t_1 \cdots t_n} \; g_{r_1\sigma_1\tau_1} \cdots g_{r_n\sigma_n\tau_n} \qquad (2.5)$$

This is a class 3 (or 3 range) determinant function of degree n. This expression has n^{3n} components corresponding to the integers assigned the virtual ranges. The value of the component of such an expression when the virtual ranges take the sequence $1 \cdots n$ is ordinarily called a *determinant* or in case π's replace δ's, a *permanent* with appropriate names for combinations. We shall not be careful, however, to make this distinction and will speak of 2.5 for example, thinking of any component, as a determinant.

3. *Elementary properties of the fundamental tensors.* The elementary properties of a tensor function are apparent in the notation.

(1) The virtual range on the multiplying tensor π or δ is permanent or alternant.

(2) Any substitution on an umbral range of elements only is equivalent to the same substitution on the virtual indices of the multiplying tensors carrying the same umbral indices.

(3) If the tensor function has a γ for a multiplying tensor the fixed range is in general neither permanent nor alternant, otherwise the fixed range is permanent or alternant according as the number of δ's having one range multiplied by inner multiplication (i. e. summed) is even or odd. The proof of the property is as follows: Consider the right member of (2.5) (to illustrate) when the r's are rearranged by a substitution S. They may be arranged in the original order by a permutation S^{-1} on the g's. But this rearranges the remaining umbral ranges by a substitution S^{-1}. The order of these, however, may be restored by renaming the umbral ranges. This will leave the umbral indices on the multiplying tensors only, permuted by substitution S. If the number of δ's is even these indices may be arranged in the original order and the determinant has been brought from one form to another with a different arrangement of indices in the fixed range without a change in value. If the number of δ's is odd there will be a change in sign if S is odd and the range is alternating. An illustration of this is found in the two range determinant

$$g_{(r_n)(s_n)} = \delta^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n} \; g_{r_1\sigma_1} \cdots g_{r_n\sigma_n} \qquad (3.1)$$

The parentheses on the range letters indicate that the range is alternating, while the brackets will indicate that the range is permanent. The indices in a fixed range need not be distinct.

(4) Alternating ranges may be interchanged. For example

$$g_{[r_n](s_n)(t_n)} = g_{[r_n](t_n)(s_n)}$$

for the number of changes of sign to bring (s_n) to (t_n) is balanced by the number to bring (t_n) to (s_n), (s_n) and (t_n) being permutations of the same integers.

4. *Permanents, Transformations of Coordinates, Inner Products.* The following combinatory properties of the permanent tensor will be found useful: *

$$\pi^{r_1 \cdots r_n}_{\rho_1 \cdots \rho_n} \; \pi^{\rho_1 \cdots \rho_n}_{s_1 \cdots s_n} = n! \pi^{r_1 \cdots r_n}_{s_1 \cdots s_n} \tag{4.1}$$

or more generally:

$$\pi^{r_1 \cdots r_p \, s_1 \cdots s_q}_{\rho_1 \cdots \rho_p \, u_1 \cdots u_q} \; \pi^{\rho_1 \cdots \rho_p}_{t_1 \cdots t_p} = p! \, \pi^{r_1 \cdots r_p \, s_1 \cdots s_q}_{t_1 \cdots t_p \, u_1 \cdots u_q} \tag{4.2}$$

The proofs follow readily from (2.1) and (2.3).

Analogous to the method of finding the equation of transformation of a determinant † the equation of transformation of a permanent may be found. For covariant elements a_{rs} the permanent is

$$\pi^{\rho_1 \cdots \rho_n}_{r_1 \cdots r_n} \; a_{\rho_1 s_1} \cdots a_{\rho_n s_n}. \tag{4.3}$$

Any interchange of the virtual ranges of the expression leaves its value unchanged, for example, interchange $(s_1 s_2)$, follow this by interchanging $(\rho_1 \rho_2)$ and then note that $\pi^{\rho_1 \rho_2 \cdots \rho_n}_{r_1 r_2 \cdots r_n} = \pi^{\rho_2 \rho_1 \cdots \rho_n}_{r_1 r_2 \cdots r_n}$ and the expression has been changed to original value without a change of value. By the definition of the tensor π (4.3) may be written

$$\sum \delta^{\rho_1}_{r_1} \cdots \delta^{\rho_n}_{r_n} \; a_{\rho_1 s_1} \cdots a_{\rho_n s_n}.$$

Consider a typical term of this expression under Σ in which the ρ's have been permuted by a substitution S. Before allowing the ρ's to be summed, permute the elements by S. Then summing with respect to the ρ's this term is $a_{r_1 \sigma_1} \cdots a_{r_n \sigma_n}$ where the σ's are a permutation S of the $s_1 \cdots s_n$. The entire expression is then

$$\pi^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n} \; a_{r_1 \sigma_1} \cdots a_{r_n \sigma_n} = \pi^{\rho_1 \cdots \rho_n}_{s_1 \cdots s_n} \; a_{r_1 \rho_1} \cdots a_{r_n \rho_n} \tag{4.4}$$

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* For the determinant tensor see reference (3), Article I.

† See Paper I.

In a similar manner we arrive at the fact that any permanent range may be fixed and the others umbral if there are an even number of δ's. This may be illustrated by

$$\pi^{(\sigma_n)}_{(s_n)}\ \pi^{(\tau_n)}_{(t_n)}\ \delta^{(\upsilon_n)}_{(u_n)}\ \delta^{(\mu_n)}_{(m_n)}\ (a_{r_n\sigma_n\tau_n\upsilon_n\mu_n}). \tag{4.11}$$

Let $\Sigma\ (\delta^{\sigma_n}_{s_n})$ be substituted for the first tensor. Consider a typical term in which the permutation of the σ's is S. Permute the a's by S. Then permute all umbral letters on the elements by S^{-1}. If there is an even number of δ's the total effect is to permute the fixed range to order $S(r_n') = (\rho_n)$ say. Summing all such terms we have

$$\pi^{(\rho_n)}_{(r_n)}\ \pi^{(\tau_n)}_{(t_n)}\ \delta^{(\upsilon_n)}_{(u_n)}\ \delta^{(\mu_n)}_{(m_n)}\ (a_{\rho_n\sigma_n\tau_n\upsilon_n\mu_n}) \tag{4.12}$$

as equivalent to (4.11). The method of changing from a fixed alternating range to a fixed permanent range will be illustrated in (5.1) and (5.2).

The transformation of a permanent. We may represent the transformed permanent by

$$P'_{r_1 \ldots r_n\, s_1 \ldots s_n} \equiv \pi'^{\rho_1 \ldots \rho_n}_{r_1 \ldots r_n}\, a'_{\rho_1 s_1} \cdots a'_{\rho_n s_n} \quad \text{which we write for brevity}$$

$$P'_{[r_n][s_n]} \equiv \pi'^{[\rho_n]}_{[r_n]}\ (a'_{\rho_n s_n}) = 1/n!\ \pi'^{[\rho_n]}_{[r_n]}\ \pi^{[\sigma_n]}_{[s_n]}\ (a'_{\rho_n\sigma_n})$$

$$= 1/n!\ \pi'^{[\rho_n]}_{[r_n]}\ \pi^{[\sigma_n]}_{[s_n]}\ a_{\alpha_n\beta_n}\ \left(\frac{\partial x_{\alpha_n}}{\partial x'_{\rho_n}}\ \frac{\partial x_{\beta_n}}{\partial x'_{\sigma_n}}\right)$$

$$= 1/n!\ \pi^{[\rho_n]}_{[r_n]}\ \left(\frac{\partial x_{\alpha_n}}{\partial x'_{\rho_n}}\right)\ \pi^{[\sigma_n]}_{[s_n]}\ \left(\frac{\partial x_{\beta_n}}{\partial x'_{\sigma_n}}\right) \cdot (a_{\alpha_n\beta_n}).$$

In this expression the $[\alpha_n]$ appearing on the partial derivatives is a range of a permanent and therefore permanent in character, so we can continue the equation

$$= 1/(n!)^2\ \pi^{[\rho_n]}_{[r_n]}\ \left(\frac{\partial x_{\gamma_n}}{\partial x'_{\rho_n}}\right)\ \pi^{[\sigma_n]}_{[s_n]}\ \left(\frac{\partial x_{\beta_n}}{\partial x'_{\sigma_n}}\right)\ \pi^{[\alpha_n]}_{[\gamma_n]}\ (a_{\alpha_n\beta_n})$$

$$= 1/(n!)^2\ K_{[r_n][\gamma_n]} \cdot K_{[s_n][\beta_n]} \cdot P_{[\gamma_n][\beta_n]}.$$

If the summation in this equation is extended only to distinct sets the numerical factor is removed. A bar under the umbral letters will indicate this convention. Thus

$$P'_{[r_n][s_n]} = K_{[r_n][\underline{\gamma_n}]} \cdot K_{[s_n][\underline{\beta_n}]} \cdot P_{[\gamma_n][\beta_n]}. \tag{4.5}$$

This equation is identical in form to equation (6.4) of paper I. There the J's represent the Jacobians or determinants of the transformation, while here the K's represent the permanents of the transformation. In (6.4) paper I

the inner product of two alternating Jacobians reduces to the square of the Jacobian. This simplification of the inner product of permanents is not possible since the absolute values of non-vanishing permanent components differ. Unlike determinants the sets of indices in brackets need not be equal.

Inner Products of Determinants and Permanents. There is a very close similarity in the inner products of these two functions.

Let $$D^{(\gamma_n)(s_n)} = \delta^{(\gamma_n)}_{(\rho_n)} \ (d^{\rho_n s_n})$$

and $$E_{(\gamma_n)(a_n)} = \delta^{(a_n)}_{(\gamma_n)} \ (e_{a_n a_n})$$

then the inner product of these will $= n!\ \delta^{(a_n)}_{(\rho_n)} \ (d^{\rho_n s_n})(e_{a_n a_n})$

$$= n! \sum_{\rho} \ (\delta^{a_n}_{\rho_n})(d^{\rho_n s_n} e_{a_n a_n}).$$

Now consider a typical term where the permutation on the ρ's is S. Permute the d's by S. This will leave the s's permuted by S. Sum the ρ's getting to within a sign

$$n!\ (d^{a_n \sigma_n} e_{a_n a_n}) \text{ where } S(s_n) = (\sigma_n).$$

The sum for all permutations S under Σ will be

$$n!\ \delta^{(s_n)}_{(\sigma_n)} \ (d^{a_n \sigma_n} e_{a_n a_n}).$$

If we set

$$f_a{}^{\sigma} = d^{a\sigma} e_{a a} \tag{4.6}$$

the inner product

$$D^{(\gamma_n)(s_n)} E_{(\gamma_n)(a_n)} = n!\ \delta^{(s_n)}_{(\sigma_n)} \ (f^{\sigma_n}_{a_n}) \ = n!\ F^{(s_n)}_{(a_n)} \tag{4.7}$$

Since the (γ_n) ranges are alternating this gives the ordinary multiplication theorem for determinants

$$D^{(r_n)(s_n)} E_{(r_n)(a_n)} = \delta^{(s_n)}_{(\sigma_n)} \ (f^{\sigma_n}_{a_n}), \ (r_n) = 1 \cdots n$$

$$= F^{(s_n)}_{(a_n)} \text{ with } (4.6). \tag{4.8}$$

For permanents, every step in the proof will hold down to (4.6) and (4.7) but the argument by which (4.8) was derived depended upon alternating indices and is not valid for permanents. We will here state the results, if P, Q and R are permanents with elements

$$p^{\rho\sigma}, \ q_{\rho\sigma} \text{ and } r_{\rho}{}^{\sigma} \quad \text{where } r_{\rho}{}^{\sigma} = p^{a\sigma} q_{a\rho} \tag{4.9}$$

then

$$P^{[\gamma_n][s_n]} \ Q_{[\gamma_n][a_n]} = n!\ R^{[s_n]}_{[a_n]} \ . \tag{4.91}$$

5. *Alternative forms for determinants. Decomposition.* There are methods of writing a determinant alternative with (2.5) for example

$$n!\, g_{[r_n](s_n)(t_n)} = \pi^{\rho_1 \cdots \rho_n}_{r_1 \cdots r_n} \; \delta^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n} \; \delta^{\tau_1 \cdots \tau_n}_{t_1 \cdots t_n} \; g_{\rho_1\sigma_1\tau_1} \cdots g_{\rho_n\sigma_n\tau_n} \qquad (5.1)$$

or

$$g_{[r_n](s_n)(t_n)} = \pi^{\rho_1 \cdots \rho_n}_{r_1 \cdots r_n} \; \delta^{\tau_1 \cdots \tau_n}_{t_1 \cdots t_n} \; g_{\rho_1 s_1 \tau_1} \cdots g_{\rho_n s_n \tau_n} \qquad (5.2)$$

Equation (5.1) is obvious when it is noticed that $[\rho_n]$ is a permanent range, there being in this case an even number of inner multiplications with δ's hence all of the $n!$ term in the sum for $[\rho_n]$ are equal to $g_{[r_n](s_n)(t_n)}$. Equation (5.2) may be proved directly or verified in this manner. By making a substitution in the (s_n) and arguing as we have above, (s_n) may be seen to be alternating. By changing s to ρ in the right member of (5.2) and multiplying by $\delta^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n}$ the member is multiplied by $n!$ since the inner product of two alternating ranges is $n!$ times the ordinary product. But this gives the right member of (5.1) which equals $n!\, g_{[r_n](s_n)(t_n)}$. By dividing by $n!$ equation 5.2 is obtained. In these equations the first range is permanent. Corresponding equations are obtained when the other ranges are fixed.

5. Cayley originally considered higher determinants as sums of two range or ordinary determinants, and methods of decomposition are current in the literature.* A single illustration will serve to show how this is done with the present notation. In the notation of (2.5) the fixed range is virtual, so the number of ranges is one greater than the number of tensor multipliers of the elements. When the expression is written

$$\delta^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n} \; [\delta^{\tau_1 \cdots \tau_n}_{t_1 \cdots t_n} \; g_{r_1\sigma_1\tau_1} \cdots g_{r_n\sigma_n\tau_n}] \qquad (5.3)$$

the expression in the brackets is a class 2 determinant since but one δ is used in expressing it. The first range in this case must be considered as being the bipartite range $r_1\sigma_1, \cdots r_n\sigma_n$. When σ is summed there will be $n!$ such determinants. Either δ may be taken out of the brackets. The extension to the types of determinants classified at the close of article 4 is apparent.

6. *Multiplication of Determinants. Demonstration of the applicability of the method to more general functions than heretofore contemplated.* The muliplication of two cubic determinants of order 2 illustrates the process. However, for file multiplication the process will be carried out in its greatest generality.

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* L. H. Rice, *American Journal of Math.*, 1918, p. 242.

Element Multiplication. For the product of the two cubic determinants g and h we have

$$g \cdot h = \delta^{\sigma_1\sigma_2}_{s_1 s_2}\ \delta^{\tau_1\tau_2}_{t_1 t_2}\ g_{r_1\sigma_1\tau_1}\ g_{r_2\sigma_2\tau_2}\ \delta^{b_1 b_2}_{\beta_1\beta_2}\ \delta^{c_1 c_2}_{\gamma_1\gamma_2}\ h^{a_1\beta_1\gamma_1}\ h^{a_2\beta_2\gamma_2}$$
$$= \delta^{\sigma_1\sigma_2}_{s_1 s_2}\ \delta^{\tau_1\tau_2}_{t_1 t_2}\ \delta^{b_1 b_2}_{\beta_1\beta_2}\ \delta^{c_1 c_2}_{\gamma_1\gamma_2}\ \kappa^{a_1\beta_1\gamma_1}_{r_1\sigma_1\tau_1} \cdot \kappa^{a_2\beta_2\gamma_2}_{r_2\sigma_2\tau_2} = \kappa \tag{6.1}$$

where $$\kappa^{a\beta\gamma}_{r\sigma\tau} = g_{r\sigma\tau}\ h^{a\beta\gamma} \tag{6.2}$$

The class of k is found by counting the δ's and adding 1. In general then for classes p and q the product k will be of class $p-1+q-1+1 = p+q-1$. In the present case r, and a are permanent indices (property 3, article 3) so can be taken identical. The point so apparent here is that in this method not a property of the δ's is utilized, we might have δ's, π's, γ's or in fact it is immaterial what they are. The class of k can be increased to $p+q$. There will by this method always be two fixed ranges, in this case the ranges r and a. The range r can be changed to $\rho_1 \cdots \rho_n$ and the determinant multiplied by $1/n!\,\pi^{\rho_1 \cdots \rho_n}_{r_1 \cdots r_n}$ where ρ is summed. If r had been alternating we should multiply by $1/n!\,\delta^{\rho_1 \cdots \rho_n}_{r_1 \cdots r_n}$ otherwise the product would vanish. Now the class will be equal to the number of ranges. Remarks such as this one on raising the class are so trivial by the analytic method that one hesitates to make them but from the geometric treatment they seem of more importance.

File Multiplication. The analytical analogue of the geometric nomenclature, "file multiplication" is "inner multiplication." Beginning with equation (6.1) and setting $t = c = \rho$ and summing (the number of choices of umbral indices is the product of the numbers of alternating ranges, each leading to a multiplication theorem) we get

$$g \cdot h = \delta^{\sigma_1\sigma_2}_{s_1 s_2}\ \delta^{b_1 b_2}_{\beta_1\beta_2}\ \delta^{\tau_1\tau_2}_{\gamma_1\gamma_2}\ g_{r_1\sigma_1\tau_1}\ g_{r_2\sigma_2\tau_2}\ h^{a_1\beta_1\gamma_1}\ h^{a_2\beta_2\gamma_2}$$
$$= \delta^{\sigma_1\sigma_2}_{s_1 s_2}\ \delta^{b_1 b_2}_{\beta_1\beta_2}\ [\delta^{\tau_1}_{\gamma_1}\delta^{\tau_2}_{\gamma_2} - \delta^{\tau_2}_{\gamma_1}\delta^{\tau_1}_{\gamma_2}]\ g_{r_1\sigma_1\tau_1}\ g_{r_2\sigma_2\tau_2}\ h^{a_1\beta_1\gamma_1}\ h^{a_2\beta_2\gamma_2}$$
$$= \delta^{\sigma_1\sigma_2}_{s_1 s_2}\ \delta^{b_1 b_2}_{\beta_1\beta_2}\ [g_{r_1\sigma_1\gamma_1}\ g_{r_2\sigma_2\gamma_2}\ h^{a_1\beta_1\gamma_1}\ h^{a_2\beta_2\gamma_2}$$
$$- g_{r_1\sigma_1\gamma_2}\ g_{r_2\sigma_2\gamma_1}\ h^{a_1\beta_1\gamma_1}\ h^{a_2\beta_2\gamma_2}]$$
$$= \delta^{\sigma_1\sigma_2}_{s_1 s_2}\ \delta^{b_1 b_2}_{\beta_1\beta_2}\ [\kappa^{a_1\beta_1}_{r_1\sigma_1}\ \kappa^{a_2\beta_2}_{r_2\sigma_2} - \kappa^{a_2\beta_2}_{r_1\sigma_1}\ \kappa^{a_1\beta_1}_{r_2\sigma_2}]$$

where $$\kappa^{a\beta}_{r\sigma} = g_{r\sigma\gamma}\ h^{a\beta\gamma} \tag{6.3}$$

$$= \delta^{\sigma_1\sigma_2}_{s_1 s_2}\ \delta^{b_1 b_2}_{\beta_1\beta_2}\ [\kappa^{a_1\beta_1}_{r_1\sigma_1}\ \kappa^{a_2\beta_2}_{r_2\sigma_2} + \kappa^{a_2\beta_1}_{r_1\sigma_1}\ \kappa^{a_1\beta_2}_{r_2\sigma_2}]$$

from the interchange of umbral indices β_1 and β_2 and use of alternating property of $\delta^{b_1 b_2}_{\beta_1\beta_2}$. Thus

$$g \cdot h = \delta^{\sigma_1\sigma_2}_{s_1 s_2}\ \delta^{b_1 b_2}_{\beta_1\beta_2}\ \pi^{a_1 a_2}_{a_1 a_2}\ \kappa^{a_1\beta_1}_{r_1\sigma_1}\ \kappa^{a_2\beta_2}_{r_2\sigma_2}$$
$$= \kappa^{[a_1 a_2](b_1 b_2)}_{[r_1 r_2](s_1 s_2)}$$

where as before we exhibit the permanent and alternating ranges by brackets and parentheses.

Multiplication of a p range and a q range tensor function. The $T_{(1)} \cdots T_{(q-1)}$ are arbitrary sets of quantities which may be permanent tensors, determinant tensors γ's or otherwise. The determinants to be multiplied are

$$A = T_{(1)} \begin{matrix} \sigma_1 \cdots \sigma_n \\ s_1 \cdots s_n \end{matrix} \cdots T_{(p-1)} \begin{matrix} \tau_1 \cdots \tau_n \\ t_1 \cdots t_n \end{matrix} \delta^{v_1 \cdots v_n}_{u_1 \cdots u_n} g_{r_1\sigma_1 \cdots \tau_1 v_1} \cdots g_{r_n\sigma_n \cdots \tau_n v_n}$$

and

$$B = \pi^{b_1 \cdots b_n}_{\beta_1 \cdots \beta_n} \cdots \pi^{c_1 \cdots c_n}_{\gamma_1 \cdots \gamma_n} \delta^{e_1 \cdots e_n}_{\epsilon_1 \cdots \epsilon_n} \cdots \delta^{f_1 \cdots f_n}_{\eta_1 \cdots \eta_n} \delta^{u_1 \cdots u_n}_{\mu_1 \cdots \mu_n} \times h^{a_1\beta_1 \cdots \gamma_1\epsilon_1} \cdots \eta_1\mu_1 \cdots h^{a_n\beta_n \cdots \gamma_n\epsilon_n \cdots \eta_n\mu_n},$$

where A contains *at least one* alternating range indicated by one δ and B contains at least one δ (but an even number of alternating ranges by property 3, art. 3) while the remaining ranges are either permanent or alternating as indicated by the π's and δ's. The total number of δ's in B will be designated by K ($\leqslant 1$). The number of π's is immaterial. The last δ in both A and B contains the indices $u_1 \cdots u_n$ which must be a permutation of $1 \cdots n$ so the product of these δ's in the product AB will be

$$\delta^{v_1 \cdots v_n}_{\mu_1 \cdots \mu_n} = \sum_{v} \delta^{v_1}_{\mu_1} \cdots \delta^{v_n}_{\mu_n} \qquad (1)$$

When this sum is multiplied with the product of the h's the sum of h's is

$$\sum_{v} h^{a_1\beta_1 \cdots \gamma_1\epsilon_1 \cdots \eta_1 v_1} \cdots h^{a_n\beta_n \cdots \gamma_n\epsilon_n \cdots \eta_n v_n} \qquad (2)$$

In (1) the sign of a term is ± 1 according as the class of the permutation of subscripts of v's is even or odd—thus in (2) the sign of a term will be ± 1 according as the permutation with respect to indices of any letter say a, is even or odd. Consider a term of this sum and let S represent the permutation of the v's in it. The v's may be brought to former order by a permutation S^{-1} of the h's which has the effect of permuting the other indices by S^{-1}. All of the umbral indices may be brought to the original order but this will leave the lower indices on the π's and δ's permuted to order S. This is of no consequence on the π's but to bring the indices of δ's to the original orders will require a permutation S^{-1} on each of $K - 1$ δ's so if $K - 1$ is even the sign of the term is unchanged, otherwise it is changed when S^{-1} is an odd permutation. The a's are in the order S^{-1} to which they were brought by the permutation on the h's. In case the sign is unchanged it is of class S which

is also of class S^{-1} which indicates that if $a_1 \cdots a_n$ is the new order of the a's the sign of the term is $\delta^{a_1 \cdots a_n}_{a_1 \cdots a_n}$. By substituting

$$C^{\alpha\beta \cdots \gamma\epsilon \cdots \eta}_{r\sigma \cdots \tau} = g^{r\sigma \cdots \tau\upsilon} h^{\alpha\beta \cdots \gamma\epsilon \cdots \eta\upsilon}$$

the product may be written

$$A \cdot B = T_{(1)} \begin{matrix}\sigma_1 \cdots \sigma_n \\ s_1 \cdots s_n\end{matrix} \cdots T_{(p-1)} \begin{matrix}\tau_1 \cdots \tau_n \\ t_1 \cdots t_n\end{matrix} \delta^{a_1 \cdots a_n}_{a_1 \cdots a_n} \pi^{b_1 \cdots b_n}_{\beta_1 \cdots \beta_n} \cdots \pi^{c_1 \cdots c_n}_{\gamma_1 \cdots \gamma_n}$$
$$\times \delta^{e_1 \cdots e_n}_{\epsilon_1 \cdots \epsilon_n} \cdots \delta^{f_1 \cdots f_n}_{\eta_1 \cdots \eta_n} \cdot C^{\alpha_1\beta_1 \cdots \gamma_1\epsilon_1 \cdots \eta_1}_{r_1\sigma_1 \cdots \tau_1} \cdots C^{\alpha_n\beta_n \cdots \gamma_n\epsilon_n \cdots \eta_n}_{r_n\sigma_n \cdots \tau_n}$$
$$= C.$$

The class of C is found by the number of T's, π's, δ's equal to $p-1+q-1$, the original number, -2 lost by inner multiplication $+1$ regained by the artifice $+1$ by which the class exceeds the umbral ranges $= p+q-2$.

The Associated Contravariant Tensor. The following method of obtaining an associated contravariant tensor shows that the existence of an associated contravariant tensor depends upon the property of an alternating tensor. Let the three indices of g_{rst} be covariant. The only possible determinant in this case, other than a permanent, is given by (2.5). The equation of transformation may be written

$$G'_{[r_n](s_n)(t_n)} = G_{[\rho_n](s_n)(t_n)} \left(\frac{\partial x_{\rho_n}}{\partial x'_{r_n}}\right) \cdot J^2 \tag{6.5}$$

So
$$\frac{\partial G'_{[r_n](s_n)(t_n)}}{\partial g'_{\rho\sigma\tau}} = J^2 \left(\frac{\partial x_{\rho_n}}{\partial x'_{r_n}}\right) \frac{\partial G_{[\rho_n](s_n)(t_n)}}{\partial g_{\alpha\beta\gamma}} \times \frac{\partial g_{\alpha\beta\gamma}}{\partial g'_{\rho\sigma\tau}}$$
$$= J^2 \left(\frac{\partial x_{\rho_n}}{\partial x'_{r_n}}\right) \frac{\partial G_{[\rho_n](s_n)(t_n)}}{\partial g_{\alpha\beta\gamma}} \cdot \frac{\partial x_\rho'}{\partial x_\alpha} \cdot \frac{\partial x'_\sigma}{\partial x_\beta} \cdot \frac{\partial x_\tau'}{\partial x_\gamma} \tag{6.6}$$

The first partial of the determinant with respect to the element has the covariant character of the determinant and an additional contravariant character. If the determinant had no permanent range, products such as $\left(\frac{\partial x_{\rho_n}}{\partial x'_{r_n}}\right)$ would not appear in the equations of transformation. There would then be an even number of alternating ranges, hence an even power of the Jacobian of the transformation would appear in (6.5) and in (6.6) so if the first partial is divided by the determinant the resulting tensor will be of contravariant character wholly. Only an element with an even number of indices, when all represent tensor character, can have an associated tensor of opposite rank. To the writer's knowledge it is only such elements that are considered as representatives of a line element in differential geometry.

A "Bibliographie des déterminants à plus de deux dimensions" to April 1924 is given by M. Lecat, Bruxelles.

Transformations Leaving Invariant Certain Partial Differential Equations of Physics.

By R. D. Carmichael.

We shall write the general wave equation in the form

$$\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = 0. \tag{1}$$

Any solution of this equation will be called a wave-function.

For the case when n is 4, the transformation $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = \sqrt{-1}\, ct$ reduces equation (1) to the classical form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

of the wave equation in theoretical physics. This is the justification for calling (1) the general wave equation.

The principal object of this paper is to determine the generators of the group of point transformations (the coordinates of the points being complex numbers) under each transformation of which a wave-function is a relative invariant. The group contains a certain linear group as a subgroup (see § 6). It is shown (see § 11) that this subgroup and two particular Cremona transformations generate the whole group when $n > 2$. When the equation is taken in the form (1), one of these Cremona transformations is real and the other is imaginary; but they are both real in the case of the classical form of the equation in mathematical physics.

It turns out that the group involved is identical with a certain group of conformal transformations (see § 3) which has been the object of several investigations. It was necessary, however, to redetermine the generators of this conformal group, since certain imaginary transformations are quite as important for our problem as the real ones which have been the object of previous investigations.

Incidental to the principal object, we have determined the corresponding groups for two other partial differential equations of physics (see § 2). (An additional equation is treated in the following paper by Mr. Goff.)

In each of the three problems, the case $n = 1$ is trivial. The case $n = 2$

(see § 7) is exceptional in character and is shown to lead to a well-known group of transformations on two variables. For this reason its exceptional features are not treated in detail.

Incidental to our main purpose we obtain new proofs of four important theorems stated below as theorems IV, V, VI, VII. The principal result is contained in theorems VIII and IX near the end of the paper.

§ 1. *The partial differential equations defining the transformations of the wave equation into itself.* Let the function $v(y_1, y_2, \cdots, y_n)$ be a solution of the wave equation

$$\sum_{k=1}^{n} \frac{\partial^2 v}{\partial y_k^2} = 0, \tag{2}$$

and let us make the transformation

$$u = w \cdot v(y_1, y_2, \cdots, y_n) \qquad w \neq 0 \tag{3}$$

where w is a function of $x_1, x_2, \cdots, x_n$ to be chosen conveniently and where $y_1, y_2, \cdots, y_n$ are taken as functionally independent functions of $x_1, x_2, \cdots, x_n$ which are to be so determined in connection with w that u shall satisfy equation (1) whenever v satisfies equation (2). Every wave-function v is thus to be transformed into a wave-function u by means of (3).

Since $1, y_i, y_i y_j$ $(i \neq j)$, $y_i^2 - y_n^2$, are solutions of (2), it follows from (3) that w and the y_i must be such that w, wy_i, wy_iy_j $(i \neq j)$, $w(y_i^2 - y_n^2)$ are wave-functions. These facts imply that $w, y_1, y_2, \cdots, y_n$ must satisfy the following system of partial differential equations:

$$\left\{\begin{array}{l} \displaystyle\sum_{k=1}^{n} \frac{\partial^2 w}{\partial x_k^2} = 0; \\ \displaystyle\sum_{k=1}^{n} \left(2 \frac{\partial w}{\partial x_k} \frac{\partial y_i}{\partial x_k} + w \frac{\partial^2 y_i}{\partial x_k^2}\right) = 0, \quad i = 1, 2, \cdots, n; \\ \displaystyle\sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} = 0, \quad i = 1, 2, \cdots, n-1, \\ \qquad\qquad\qquad\qquad j = i+1, i+2, \cdots, n; \\ \displaystyle\sum_{k=1}^{n} (\partial y_i/\partial x_k)^2 = \sum_{k=1}^{n} (\partial y_n/\partial x_k)^2, \quad i = 1, 2, \cdots, n-1. \end{array}\right. \tag{4}$$

From relation (3) we have

$$(5)\qquad \frac{\partial^2 u}{\partial x_k^2} = v\frac{\partial^2 w}{\partial x_k^2} + 2\frac{\partial w}{\partial x_k}\sum_{i=1}^{n}\frac{\partial v}{\partial y_i}\frac{\partial y_i}{\partial x_k}$$

$$+ w\left\{\sum_{i=1}^{n}\frac{\partial^2 v}{\partial y_i^2}\left(\frac{\partial y_i}{\partial x_k}\right)^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\partial^2 v}{\partial y_i \partial y_j}\frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_k} + \sum_{i=1}^{n}\frac{\partial v}{\partial y_i}\frac{\partial^2 y_i}{\partial x_k^2}\right\},$$

$$k = 1, 2, \cdots, n.$$

From these equations it is easy to see that u, as defined by (3), is a solution of (1) provided that equations (4) are satisfied.

Let us use ρ for the common value of the n sums

$$\rho = \sum_{k=1}^{n}(\partial y_i/\partial x_k)^2, \qquad i = 1, 2, \cdots, n.$$

If we employ δ_{ij} to stand for 1 or 0 according as j is or is not equal to i, then the conditions in the last two lines of (4) may be put in the form

$$(6)\qquad \sum_{k=1}^{n}\frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_k} = \delta_{ij}\rho \qquad i, j = 1, 2, \cdots, n.$$

Since the Jacobian (functional determinant)

$$(7)\qquad J = \left|\frac{\partial y_i}{\partial x_k}\right|$$

is different from zero (owing to the assumed functional independence of the y's) is follows from (6) that ρ can not be identically zero.

The results so far obtained may be put into the form of the following theorem:

THEOREM I. *Let $w, y_1, y_2, \cdots, y_n$ be a set of functions of $x_1, x_2, \cdots, x_n$ such that $y_1, y_2, \cdots, y_n$ are functionally independent and $w \not\equiv 0$. Then a necessary and sufficient condition that every solution of (2) shall be transformed through (3) into a solution of (1) is that the functions $w, y_1, y_2, \cdots, y_n$ shall satisfy the following conditions:—a function ρ exists, not identically zero, such that*

$$(8)\qquad \sum_{k=1}^{n}\frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_k} = \delta_{ij}\rho,$$

and the equations

$$(9)^* \qquad \sum_{k=1}^{n} \left(2\frac{\partial w}{\partial x_k}\frac{\partial y_i}{\partial x_k} + w\frac{\partial y^2{}_i}{\partial x_k{}^2}\right) = 0,$$

$$(10) \qquad \sum_{k=1}^{n} \frac{\partial^2 w}{\partial x_k{}^2} = 0,$$

are verified.

The totality of transformations (3), subject to the conditions just named, obviously form a group.

Corollary. A necessary and sufficient condition equivalent to that in theorem I is that each of the following functions shall be a wave-function: w; wy_i, $i = 1, 2, \cdots, n$; wy_iy_j, $i = 1, 2, \cdots, n-1$, $j = i+1, \cdots, n$; $w(y_i{}^2 - y_n{}^2)$, $i = 1, 2, \cdots, n-1$.

§ 2. *Certain related groups of transformations.* The method used in the preceding section may be employed in the proof of some results related to theorem I.

THEOREM II. *Let* $w, y_1, y_2, \cdots, y_n$ *be a set of functions of* $x_1, x_2, \cdots, x_n$ *such that* $y_1, y_2, \cdots, y_n$ *are functionally independent and* $w \not\equiv 0$. *Then a necessary and sufficient condition that every solution of the equation*

$$(11) \qquad \sum_{k=1}^{n} \frac{\partial^2 v}{\partial y_k{}^2} + v = 0$$

shall be transformed through the relation

$$(12) \qquad u = w \cdot v(y_1, y_2, \cdots, y_n)$$

into a solution of the equation

$$(13) \qquad \sum_{k=1}^{n} \frac{\partial u^2}{\partial x_k{}^2} + u = 0$$

is that the functions $w, y_1, y_2, \cdots, y_n$ *shall satisfy the following conditions: a function* ρ *exists, not identically zero, such that*

$$(14) \qquad \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_k} = \delta_{ij}\rho$$

and the equations

$$(15)^\dagger \qquad \sum_{k=1}^{n} \left(2\frac{\partial w}{\partial x_k}\frac{\partial y_i}{\partial x_k} + w\frac{\partial^2 y_i}{\partial x_k{}^2}\right) = 0$$

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* In § 5, these conditions are replaced by the simpler one, $w = \rho^{\frac{n-2}{4}}$. See also the latter part of § 5.

† See the preceding footnote.

$$\sum_{k=1}^{n} \frac{\partial^2 w}{\partial x_k^2} + w = \rho w \tag{16}$$

are satisfied.

The necessity of these conditions is easily proved by aid of the following solutions of (11): $\sin y_i$ and $\cos y_i$ (or $e^{y_i\sqrt{-1}}$ and $e^{-y_i\sqrt{-1}}$) taken in conjunction, and $\sin (y_i\sqrt{s}) \sin (y_j\sqrt{1-s})$ where $i \neq j$ and s is a constant different from 0 and 1. In order to show that the conditions are sufficient one may proceed as in the proof of the corresponding part of theorem I.

The partial differential equations of the group of transformations x_i into y_i and u into v such that u, as defined by the transformation $u = w \cdot v(y_1, y_2, \cdots, y_n)$, shall be a solution of the equation

$$\sum_{k=1}^{n} (\partial u/\partial x_k)^2 = 0 \tag{17}$$

whenever $v(y_1, y_2, \cdots, y_n)$ is a solution of the equation

$$\sum_{k=1}^{n} (\partial v/\partial y_k)^2 = 0 \tag{18}$$

may be determined in a similar manner. The following sufficient conditions are readily obtained:

$$\sum_{k=1}^{n} (\partial w/\partial x_k)^2 = 0;$$

$$\sum_{k=1}^{n} \frac{\partial w}{\partial x_k} \frac{\partial y_i}{\partial x_k} = 0, \quad i = 1, 2, \cdots, n;$$

$$\sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} = 0, \quad i \neq j;$$

$$\sum_{k=1}^{n} (\partial y_i/\partial x_k)^2 = \sum_{k=1}^{n} (\partial y_n/\partial x_k)^2 \neq 0, \qquad i = 1, 2, \cdots, n-1.$$

That these conditions are also necessary may be shown by means of the following solutions of (18): $v = 1$, $v = f(y_i + y_j\sqrt{-1})$, $v = f(y_i - y_j\sqrt{-1})$, $(i \neq j)$, $f(t)$ being an analytic function of t.

On multiplying the second of the foregoing equations by $\partial y_i/\partial x_l$ and summing as to i from 1 to n, we find (by aid of equation (21) in the next section) that $\partial w/\partial x_l = 0$, $l = 1, 2, \cdots, n$, so that w is a constant. Without loss of generality we may take it equal to unity; and this we do.

Then we have the following result:

THEOREM III. *A necessary and sufficient condition that every solution of* (18) *shall be transformed by the relation* $u = v(y_1, y_2, \cdots, y_n)$ *into a solution of* (17), *the functions* $y_1, y_2, \cdots, y_n$ of $x_1, x_2, \cdots, x_n$ *being functionally independent, is that a funtcion* ρ *shall exist, not identically zero, such that*

$$\sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} = \delta_{ij}\rho. \tag{19}$$

§ 3. *The transformation group of theorem III.* The transformation group of theorem III, transforming $x_1, x_2, \cdots, x_n$ into $y_1, y_2, \cdots, y_n$ contains the corresponding transformation groups of theorems I and II. We shall therefore treat first the transformation group of theorem III.

Let J denote the functional determinant

$$J = \left| \frac{\partial y_i}{\partial x_j} \right| .$$

Then by the multiplication theorem for determinants, we have

$$J^2 = \left| \frac{\partial y_i}{\partial x_j} \right| \cdot \left| \frac{\partial y_j}{\partial x_i} \right| = \left| \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} \right| = \rho^n,$$

the last member having been obtained by use of (19). For J we have therefore the value

$$J = \rho^{n/2}. \tag{20}$$

Since $J \not\equiv 0$, it follows that ρ can not be identically zero.

If we multiply both members of equation (19) by $\partial y_j/\partial x_l$ and sum as to j from 1 to n, we have

$$\sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \left(\sum_{j=1}^{n} \frac{\partial y_j}{\partial x_k} \frac{\partial y_j}{\partial x_l} \right) = \rho \frac{\partial y_i}{\partial x_l} , \quad i, l = 1, 2, \cdots, n.$$

On holding l fixed and allowing k and i to vary, we have a system of linear equations in the n quantities inclosed in parenthesis. The determinant of the coefficients of this system is J and is therefore different from zero. On solving the foregoing system for the indicated quantities, we have

$$\sum_{j=1}^{n} \frac{\partial y_j}{\partial x_k} \frac{\partial y_j}{\partial x_l} = \delta_{kl}\rho, \qquad k, l = 1, 2, \cdots, n. \tag{21}$$

Thus equations (19) imply equations (21); it may be shown similarly that equations (21) imply equations (19).

If we take $y_1, y_2, \cdots, y_n$ to be the coordinates of a Euclidean space, then the element ds of arc length is given by the equation

$$ds^2 = dy_1^2 + dy_2^2 + \cdots + dy_n^2.$$

But
$$dy_m = \sum_{k=1}^{n} \frac{\partial y_m}{\partial x_k}\, dx_k, \qquad m = 1, 2, \cdots, n.$$

Substituting these values of the dy_m into the preceding equation and simplifying by the aid of (19), we have

$$(22) \quad ds^2 = dy_1^2 + dy_2^2 + \cdots + dy_n^2 = \rho(dx_1^2 + dx_2^2 + \cdots + dx_n^2).$$

Hence the mapping effected by our transformation is conformal in character. Conversely, if a transformation of the y's into the x's is such that equation (22) is satisfied, then it may readily be shown that equations (19) are satisfied. Hence the transformation group involved in theorem III is the group of conformal point transformations in a Euclidean space of n dimensions, the coordinates of the points being complex numbers.

§ 4. *Determination of ρ for the conformal transformations.* If we consider the equations

$$x_i = x_i(y_1, y_2, \cdots, y_n)$$

of the inverse transformation, then on differentiating with respect to x_j we have

$$\delta_{ij} = \sum_{k=1}^{n} \frac{\partial x_i}{\partial y_k}\, \frac{\partial y_k}{\partial x_j}\ .$$

Multiplying by $\partial y_m/\partial x_j$, summing as to j from 1 to n and reducing by aid of (19) we have

$$(23) \qquad \frac{\partial y_m}{\partial x_i} = \rho\, \frac{\partial x_i}{\partial y_m}, \quad i, m = 1, 2, \cdots, n.$$

These equations are useful in considering the relation of a transformation to its inverse.

From equations (19) or (21) we may obtain by elimination a system of equations involving a single one of the functions y_i alone. For this purpose we have from (19) by differentiation the equations

$$\sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k}\, \frac{\partial^2 y_j}{\partial x_k \partial x_l} + \sum_{k=1}^{n} \frac{\partial y_j}{\partial x_k}\, \frac{\partial^2 y_i}{\partial x_k \partial x_l} = \delta_{ij}\, \frac{\partial \rho}{\partial x_l}\ .$$

On multiplying by $\partial y_i/\partial x_l$, summing as to i from 1 to n, and simplifying by aid of (21), we have

$$\rho \frac{\partial^2 y_j}{\partial x_l^2} + \sum_{k=1}^{n} \frac{\partial y_j}{\partial x_k} \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_l} \frac{\partial^2 y_i}{\partial x_k \partial x_l} = \frac{\partial \rho}{\partial x_l} \frac{\partial y_j}{\partial x_l}.$$

The second term of this equation may be simplified by means of the relation

$$\sum_{i=1}^{n} \frac{\partial y_i}{\partial x_l} \frac{\partial^2 y_i}{\partial x_l \partial x_k} = \tfrac{1}{2} \frac{\partial \rho}{\partial x_k}$$

obtained by replacing k by l in (21) and differentiating both members of the resulting equation with respect to x_k; thus we have

$$\rho \frac{\partial^2 y_j}{\partial x_l^2} - \frac{\partial \rho}{\partial x_l} \frac{\partial y_j}{\partial x_l} + \tfrac{1}{2} \sum_{k=1}^{n} \frac{\partial \rho}{\partial x_k} \frac{\partial y_j}{\partial x_k} = 0, \qquad l = 1, 2, \cdots, n. \tag{24}$$

From (24) may be derived some consequences which will be of use later. If we sum as to l from 1 to n we have

$$\rho \sum_{l=1}^{n} \frac{\partial^2 y_j}{\partial x_l^2} + \frac{n-2}{2} \sum_{k=1}^{n} \frac{\partial \rho}{\partial x_k} \frac{\partial y_j}{\partial x_k} = 0. \tag{25}$$

For the case when $n = 2$ this shows that y_1 and y_2 are both wave-functions.

On multiplying both members of (24) by $\partial y_j/\partial x_m$ and summing as to j we have

$$\sum_{j=1}^{n} \frac{\partial y_j}{\partial x_m} \frac{\partial^2 y_j}{\partial x_l^2} = \delta_{lm} \frac{\partial \rho}{\partial x_l} - \frac{1}{2} \frac{\partial \rho}{\partial x_m}. \tag{26}$$

On summing both members of this equation as to l we have

$$\sum_{l=1}^{n} \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_m} \frac{\partial^2 y_j}{\partial x_l^2} = \frac{2-n}{2} \frac{\partial \rho}{\partial x_m}. \tag{27}$$

We may now determine * explicitly the possible values of ρ. For this purpose it is convenient to write

$$\rho = 1/\sigma^2. \tag{28}$$

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* This is based on the work of R. Beez, *Zeitschrift für Mathematik und Physik*, 20 (1875), 253-270; it is necessary to repeat the argument for the sake of some additional formulae not given by Beez. Some simplifications in the computation are also effected. Moreover, in our problem, we have to examine a case not treated by Beez.

If then we substitute into (19) and (21) the values of $\partial y_i/\partial x_k$ and $\partial y_j/\partial x_k$ as given in (23) we have

$$(29) \qquad \sum_{k=1}^{n} \frac{\partial x_k}{\partial y_i}\,\frac{\partial x_k}{\partial y_j} = \delta_{ij}\sigma^2, \quad \sum_{k=1}^{n} \frac{\partial x_i}{\partial y_k}\,\frac{\partial x_j}{\partial y_k} = \delta_{ij}\sigma^2.$$

Differentiate both members of the last equation with respect to y_l and in the result replace $\partial x_j/\partial y_k$ and $\partial x_i/\partial y_k$ by their values as given by (23) and simplify; thus we have

$$\sum_{k=1}^{n} \frac{\partial^2 x_i}{\partial y_k \partial y_l}\,\frac{\partial y_k}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial^2 x_j}{\partial y_k \partial y_l}\,\frac{\partial y_k}{\partial x_i} = \delta_{ij}\,\frac{2}{\sigma}\,\frac{\partial \sigma}{\partial y_l};$$

or

$$(30) \qquad \frac{\partial}{\partial x_j}\left(\frac{\partial x_i}{\partial y_l}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial x_j}{\partial y_l}\right) = \delta_{ij}\,\frac{1}{\sigma}\,\frac{\partial \sigma}{\partial y_l}$$

From the evident relation

$$\frac{\partial}{\partial x_j}\left(\frac{\partial y_l}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial y_l}{\partial x_j}\right)$$

we have by means of (23) the relation

$$\frac{\partial}{\partial x_j}\left(\frac{1}{\sigma^2}\,\frac{\partial x_i}{\partial y_l}\right) = \frac{\partial}{\partial x_i}\left(\frac{1}{\sigma^2}\,\frac{\partial x_j}{\partial y_l}\right);$$

or,

$$\frac{\partial}{\partial x_j}\left(\frac{\partial x_i}{\partial y_l}\right) - \frac{2}{\sigma}\,\frac{\partial x_i}{\partial y_l}\,\frac{\partial \sigma}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial x_j}{\partial y_l}\right) - \frac{2}{\sigma}\,\frac{\partial x_j}{\partial y_l}\,\frac{\partial \sigma}{\partial x_i}.$$

Combining this with (30) we have the relation

$$(31) \qquad \frac{\partial}{\partial x_j}\left(\frac{\partial x_i}{\partial y_l}\right) = (1/\sigma)\left(\frac{\partial x_i}{\partial y_l}\,\frac{\partial \sigma}{\partial x_j} - \frac{\partial x_j}{\partial y_l}\,\frac{\partial \sigma}{\partial x_i} + \delta_{ij}\,\frac{\partial \sigma}{\partial y_l}\right).$$

Differentiating with respect to x_m we have

$$\begin{aligned}\frac{\partial^2}{\partial x_m \partial x_j}\left(\frac{\partial x_i}{\partial y_l}\right) = (1/\sigma)\Big\{ & \frac{\partial}{\partial x_m}\left(\frac{\partial x_i}{\partial y_l}\right)\frac{\partial \sigma}{\partial x_j} \\ & + \frac{\partial x_i}{\partial y_l}\,\frac{\partial^2 \sigma}{\partial x_m \partial x_j} - \frac{\partial}{\partial x_m}\left(\frac{\partial x_j}{\partial y_l}\right)\frac{\partial \sigma}{\partial x_i} - \frac{\partial x_j}{\partial y_l}\,\frac{\partial^2 \sigma}{\partial x_m \partial x_i} \\ & + \delta_{ij}\,\frac{\partial}{\partial x_m}\left(\frac{\partial \sigma}{\partial y_l}\right)\Big\} - \frac{1}{\sigma^2}\,\frac{\partial \sigma}{\partial x_m}\left(\frac{\partial x_i}{\partial y_l}\,\frac{\partial \sigma}{\partial x_j} - \frac{\partial x_j}{\partial y_l}\,\frac{\partial \sigma}{\partial x_i}\right. \\ & \left. + \tfrac{1}{2}\,\delta_{ij}\,\frac{\partial \sigma}{\partial y_l}\right).\end{aligned}$$

For the first and third terms in the brace { } substitute their values as given by (31); on simplifying we have the following equation:

$$(32)\qquad \frac{\partial^2}{\partial x_m \partial x_j}\left(\frac{\partial x_i}{\partial y_l}\right) = 1/\sigma\left(\frac{\partial x_i}{\partial y_l}\frac{\partial^2\sigma}{\partial x_m \partial x_j} - \frac{\partial x_j}{\partial y_l}\frac{\partial^2\sigma}{\partial x_m \partial x_i} + \delta_{ij}\frac{\partial}{\partial x_m}\frac{\partial\sigma}{\partial y_l}\right) + \frac{1}{\sigma^2}\frac{\partial\sigma}{\partial y_l}\left(\delta_{im}\frac{\partial\sigma}{\partial x_j} - \delta_{jm}\frac{\partial\sigma}{\partial x_i} - \delta_{ij}\frac{\partial\sigma}{\partial x_m}\right).$$

On interchanging m and j and comparing the resulting equation with (32) itself we see that

$$(33)\qquad \frac{\partial x_m}{\partial y_l}\frac{\partial^2\sigma}{\partial x_j \partial x_i} - \frac{\partial x_j}{\partial y_l}\frac{\partial^2\sigma}{\partial x_m \partial x_i} - \delta_{im}\frac{\partial}{\partial x_j}\left(\frac{\partial\sigma}{\partial y_l}\right) + \delta_{ij}\frac{\partial}{\partial x_m}\left(\frac{\partial\sigma}{\partial y_l}\right) = \frac{2}{\sigma}\frac{\partial\sigma}{\partial y_l}\left(\delta_{ij}\frac{\partial\sigma}{\partial x_m} - \delta_{im}\frac{\partial\sigma}{\partial x_j}\right).$$

When $n > 2$ it is possible to choose i, j, m all different; if we do so we have

$$\frac{\partial x_m}{\partial y_l}\frac{\partial^2\sigma}{\partial x_j \partial x_i} = \frac{\partial x_j}{\partial y_l}\frac{\partial^2\sigma}{\partial x_m \partial x_i}, \quad n > 2,\ i, j, m \text{ all different.}$$

Multiplying by $\partial x_m/\partial y_l$ and summing as to l we see that

$$(34)\qquad \frac{\partial^2\sigma}{\partial x_j \partial x_i} = 0, \qquad i \neq j, \qquad n > 2.$$

From this it follows, when $n > 2$, that $\partial\sigma/\partial x_i$ is independent of each of the x's except x_i. Hence we have

$$(35)\qquad \sigma = X_1(x_1) + X_2(x_2) + \cdots + X_n(x_n), \qquad n > 2,$$

where $X_i(x_i)$ is a function of x_i alone.

In (33) take $j = i$ and $m \neq i$; then we have

$$(36)\qquad \frac{\partial x_m}{\partial y_l}\frac{\partial^2\sigma}{\partial x_i^2} = \frac{2}{\sigma}\frac{\partial\sigma}{\partial y_l}\frac{\partial\sigma}{\partial x_m} - \frac{\partial}{\partial x_m}\left(\frac{\partial\sigma}{\partial y_l}\right), \qquad m \neq i.$$

For each m there exists a y_l such that $\partial x_m/\partial y_l$ is not identically zero. Taking $m = 1$ we see that $\partial^2\sigma/\partial x_i^2$ is independent of i if i belongs to the set $2, 3, \cdots, n$. Then taking $m = 2$ we see that $\partial^2\sigma/\partial x_i^2$ is independent of i if i belongs to the set $1, 3, 4, \cdots, n$. Hence when $n > 2$ it is independent of i for $i = 1, 2, \cdots, n$. But from (35) it follows that $\partial^2\sigma/\partial x_i^2$ is a function of x_i alone. It must therefore be a constant. Calling this constant $2c$ we may therefore write

$$(37)\qquad \frac{\partial^2\sigma}{\partial x_i^2} = 2c, \qquad i = 1, 2, \cdots, n, \qquad n > 2.$$

Before undertaking to evaluate σ completely it is desirable to obtain still another simple restrictive condition on it. In (32) take $j = i$, $m \neq i$, $n > 2$, and simplify by aid of (34); thus we have

$$\frac{\partial^2}{\partial x_i \partial x_m}\left(\frac{\partial x_i}{\partial y_l}\right) = \frac{1}{\sigma}\frac{\partial}{\partial x_m}\left(\frac{\partial \sigma}{\partial y_l}\right) - \frac{1}{\sigma^2}\frac{\partial \sigma}{\partial x_m}\frac{\partial \sigma}{\partial y_l}, \qquad m \neq i, \quad n > 2.$$

In (32) take $m = i$, $j \neq i$, $n > 2$, and simplify by use of (34); thus we have

$$\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{\partial x_i}{\partial y_l}\right) = -\frac{1}{\sigma}\frac{\partial x_j}{\partial y_l}\frac{\partial^2 \sigma}{\partial x_i^2} + \frac{1}{\sigma^2}\frac{\partial \sigma}{\partial y_l}\frac{\partial \sigma}{\partial x_j}, \qquad j \neq i, \quad n > 2.$$

In the last equation change j to m, subtract the resulting equation member by member from the preceding one, and simplify; then by the aid of (34) we have

$$\begin{aligned}
\frac{2}{\sigma}\frac{\partial \sigma}{\partial y_l}\frac{\partial \sigma}{\partial x_m} - \frac{\partial x_m}{\partial y_l}\frac{\partial^2 \sigma}{\partial x_i^2} &= \frac{\partial}{\partial x_m}\left(\frac{\partial \sigma}{\partial y_l}\right) \\
&= \frac{\partial}{\partial x_m}\sum_{k=1}^{n}\frac{\partial \sigma}{\partial x_k}\frac{\partial x_k}{\partial y_l} \\
&= \frac{\partial^2 \sigma}{\partial x_m^2}\frac{\partial x_m}{\partial y_l} + \sum_{k=1}^{n}\frac{\partial \sigma}{\partial x_k}\frac{\partial}{\partial x_m}\left(\frac{\partial x_k}{\partial y_l}\right) \\
= \frac{\partial 2\sigma}{\partial x_m^2}\frac{\partial x_m}{\partial y_l} + \frac{1}{\sigma}\frac{\partial \sigma}{\partial x_m}\sum_{k=1}^{n}\frac{\partial \sigma}{\partial x_k}\frac{\partial x_k}{\partial y_l} - \frac{1}{\sigma}\frac{\partial x_m}{\partial y_l}\sum_{k=1}^{n}(\partial\sigma/\partial x_k)^2 \\
&\qquad + \frac{1}{\sigma}\frac{\partial \sigma}{\partial x_m}\frac{\partial \sigma}{\partial y_l},
\end{aligned}$$

the last line being obtained by aid of (31). Hence, since $\partial x_m/\partial y_l$ is certainly not zero for every l, we have by aid of (37)

$$(38) \qquad \sum_{k=1}^{n}(\partial\sigma/\partial x_k)^2 = \frac{\partial^2 \sigma}{\partial x_i^2} + \frac{\partial^2 \sigma}{\partial x_m^2} = 4c\sigma, \qquad n > 2.$$

For determining σ there are two cases to be examined. Suppose first that c is different from zero. Then from (35), (37), and (38), it follows readily that σ has the form

$$(39) \qquad \sigma = c\sum_{k=1}^{n}(x_k - a_k)^2, \qquad c \neq 0, \quad n > 2,$$

where $a_1, a_2, \cdots, a_n$ are constants. Later we shall show that this value of σ does indeed belong to a possible transformation.

Suppose * next that $c = 0$. Then from (37) and (35) it follows that σ is a linear function of $x_1, x_2, \cdots, x_n$. If we write

$$\sigma = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n + \beta, \qquad n > 2, \tag{40}$$

we see from (38) that we must have

$$\beta_1^2 + \beta_2^2 + \cdots + \beta_n^2 = 0. \tag{41}$$

The transformations which arise for such values of σ will be considered later. In particular we must consider the case when σ is a constant, that is, when $\beta_1, \beta_2, \cdots, \beta_n$ are all zero.

§ 5. *The determination of w.* By means of conditions (8) and (9) in theorem I or the equivalent conditions (14) and (15) in theorem II it is possible to find the value of w explicitly in terms of ρ. For this purpose we multiply both members of (9) by $\partial y_i/\partial x_l$, sum as to i from 1 to n, simplify by aid of (21) and reduce; thus we have

$$\frac{1}{w}\frac{\partial w}{\partial x_l} = -\frac{1}{2\rho}\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_l}\sum_{k=1}^{n}\frac{\partial^2 y_i}{\partial x_k^2}, \qquad l = 1, 2, \cdots, n.$$

On simplifying this equation by aid of (27) we have

$$\frac{\partial}{\partial x_l}\log w = -\frac{1}{2\rho}\,\frac{2-n}{2}\,\frac{\partial \rho}{\partial x_l} = \frac{n-2}{4}\,\frac{\partial}{\partial x_l}(\log \rho), \quad i = 1, 2, \cdots, n.$$

Hence except for a trivial constant factor (taken equal to unity) we see from this equation and (28) that

$$w = \rho^{\frac{n-2}{4}} = \sigma^{\frac{2-n}{2}}. \tag{42}$$

Thus w is completely determined in terms of ρ except for a constant factor. When $n = 2$ we have $w = 1$.

We shall now show that this value of w may be used to replace equation (9) in theorem I and equation (15) in theorem II. For this purpose it is sufficient to show that (9) is implied by (8) and (42). In this proof we may use equations (21) to (27) since they are all implied by (8). Differentiating both members of (42) logarithmically we obtain the equation preceding (42); thence by the aid of (27) we pass to the first equation of this section. On multiplying both members of that equation by $\partial y_j/\partial x_l$ and summing as to

* The case $c = 0$ is not treated by Beez in the article already cited.

l, we obtain an equation which reduces readily to the form of (9). Hence, *in theorems I and II equations* (9) *and* (15) *may each be replaced by* (42).

Equation (42) gives the value of w for the transformation groups involved in theorems I and II. In the case of theorem III we saw that the corresponding quantity w must be a constant, whence it was taken to be unity. When $n = 2$ it is unity even in the cases of theorems I and II. When w is unity the wave function is an absolute invariant in the usual sense of the term in the theory of invariants. The same is true for solutions of equation (13). Also we have absolute invariants in all cases when $\rho = 1$. In general, the solutions of equations (1) and (13) are relative invariants, the multiplying factor w being a power of the Jacobian J of the transformation, since as we see from (20) and (42) we have

$$w = J^{\frac{n-2}{2n}} . \tag{43}$$

By aid of the values of σ given in (39) and (40) it is seen that w, as defined by (42), satisfies equation (10). From this it follows that conditions (8) and (9) in theorem I are superfluous, except as they are involved in finding the value of w in terms of ρ. Hence *the groups involved in theorems I and III are the same.* With the given possible values of σ, and hence of ρ, and by aid of (42), it is seen from equation (16) that ρ must be unity in the case of the group in theorem II.

§ 6. *The case when ρ is a constant C.* When ρ is a constant C, whence σ is a constant, the second member of equation (31) is zero. Hence with the aid of (23) we see that

$$0 = \frac{\partial}{\partial x_j}\left(\frac{\partial x_i}{\partial y_l}\right) = \frac{\partial}{\partial x_j}\left(\frac{1}{c}\frac{\partial y_l}{\partial x_i}\right) = \frac{1}{c}\frac{\partial^2 y_l}{\partial x_i \partial x_j} .$$

From this it follows that each function y_l is linear in the variables $x_1, x_2, \cdots, x_n$. Then we must have y_j of the form

$$y_j = a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n + a_j, \qquad j = 1, 2, \cdots, n. \tag{44}$$

Necessary and sufficient conditions that equations (19) are satisfied are then the following:

$$\sum_{k=1}^{n} a_{ik}a_{jk} = \delta_{ij}C. \tag{45}$$

When these conditions are satisfied we have in (44) transformations belonging

to the groups involved in theorems I and III. In order that the transformations shall belong to the group involved in theorem II it is necessary to add the further condition that C shall be unity, as one sees by the aid of equation (16). In view of the result at the end of § 5 it is then seen that the transformations (44), with the named conditions, constitute the whole of the group defined in theorem III.

§ 7. *The case when* $n = 2$. When $n = 2$ it follows from equation (42) that the function w of theorem I is unity. Hence, when $n = 2$, the transformation groups of theorems I and III coincide and each of them is the group of conformal transformations in the Euclidean plane. This group is sufficiently well known to justify us in not treating it further. The results given in the next section, however, are valid for all values of n.

§ 8. *On the class of possible solutions when* ρ *is given.* From (23), (19) and (21) we see that

$$\sum_{k=1}^{n} \frac{\partial x_i}{\partial y_k} \frac{\partial x_j}{\partial y_k} = (1/\rho^2) \sum_{k=1}^{n} \frac{\partial y_k}{\partial x_i} \frac{\partial y_k}{\partial x_j} = (1/\rho^2) \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} = \delta_{ij}/\rho.$$

Hence the value of ρ for a given transformation is the reciprocal of its value for the inverse transformation.

Again, if we have a second transformation of the y's into z's we have an equation of the form

$$\sum_{k=1}^{n} \frac{\partial z_i}{\partial y_k} \frac{\partial z_j}{\partial y_k} = \delta_{ij} \bar{\rho}.$$

Then we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{\partial z_i}{\partial x_k} \frac{\partial z_j}{\partial x_k} &= \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{\partial z_i}{\partial y_l} \frac{\partial y_l}{\partial x_k} \frac{\partial z_j}{\partial y_m} \frac{\partial y_m}{\partial x_k} \\
&= \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{\partial z_i}{\partial y_l} \frac{\partial z_j}{\partial y_m} \sum_{k=1}^{n} \frac{\partial y_l}{\partial x_k} \frac{\partial y_m}{\partial x_k} \\
&= \rho \sum_{m=1}^{n} \frac{\partial z_i}{\partial y_m} \frac{\partial z_j}{\partial y_m} \\
&= \delta_{ij} \rho \bar{\rho}.
\end{aligned}$$

Hence if two transformations have the characteristic functions ρ and $\bar{\rho}$ then the product of the two have the characteristic function $\rho\bar{\rho}$.

Now let S and T be two transformations with the same characteristic function ρ. Then ρ^{-1} is the characteristic function of the inverse S^{-1} of S. Hence the transformation $S^{-1}T$ has the characteristic function 1. If we write

$$S^{-1}T = I$$

then I is a transformation with the characteristic function 1. But from the last equation it follows that $T = SI$. Hence if two transformations have the same characteristic function ρ then either one of them is equal to the other multiplied by a transformation with the characteristic function 1. In § 6 we have already determined all such transformations and have seen that they are linear. It follows that if we have a particular transformation associated with a given characteristic function ρ we can determine at once all the transformations having the same characteristic function. Therefore it will be sufficient henceforth to determine only a single transformation having a particular characteristic function ρ.

§ 9. *The case when σ is a quadratic function and $n > 2$.* The necessary form of σ in this case is given by equation (39). It remains to construct a particular transformation associated with this value of σ. Now we have

$$(46) \qquad \sum_{l=1}^{n} \left(\frac{\partial(1/\sigma)}{\partial y_l}\right)^2 = (1/\sigma^4) \sum_{l=1}^{n} \left(\frac{\partial \sigma}{\partial y_l}\right)^2 = (1/\sigma^4) \sum_{l=1}^{n} \left(\sum_{k=1}^{n} \frac{\partial \sigma}{\partial x_k} \frac{\partial x_k}{\partial y_l}\right)^2$$

$$= (1/\sigma^4) \sum_{l=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial \sigma}{\partial x_k}\right)^2 \left(\frac{\partial x_k}{\partial y_l}\right)^2 + (1/\sigma^4) \sum_{\substack{l=1 \\ k \neq m}}^{n} \sum_{k,m=1}^{n} \frac{\partial \sigma}{\partial x_k} \frac{\partial \sigma}{\partial x_m} \frac{\partial x_k}{\partial y_l} \frac{\partial x_m}{\partial y_l}$$

$$= (1/\sigma^2) \sum_{k=1}^{n} \left(\frac{\partial \sigma}{\partial x_k}\right)^2 = 4c/\sigma,$$

the last two members being gotten by aid of (23), (21) and (38). On comparing with the paragraph containing equation (39) we see that constants β must exist such that

$$1/\sigma = c \sum_{k=1}^{n} (y_k - \beta_k)^2.$$

Comparing with (39) we see that we must have

$$\sum_{k=1}^{n} (x_k - \alpha_k)^2 \cdot \sum_{k=1}^{n} (y_k - \beta_k)^2 = 1/c^2.$$

This suggests the inversion transformation with respect to an n-dimensional sphere and hence the solution

$$y_i - \beta_i = \frac{x_i - a_i}{c\{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2\}}, \qquad i = 1, 2, \cdots, n.$$

It is readily verified that these values of the y's satisfy equation (19) for the given value of σ. Hence they afford a particular transformation associated with the given value of σ. The remaining transformations associated with σ are then determined by means of the results in § 8.

On taking appropriate translations in the x-space and the y-space respectively and adopting a convenient magnification in one or both spaces we may reduce the preceding transformation to the case in which $c = 1$ and the α's and β's are all zero. This change can be effected by means of linear transformations belonging to our group. Hence, except for suitable linear transformations, the preceding transformation may be written in the form

$$y_i = x_i/s^2 \qquad \text{where } s^2 = x_1^2 + x_2^2 + \cdots + x_n^2. \tag{47}$$

This is a real Cremona transformation of period two. It leads us to the following known theorems * (in the first case by the aid of the value of w given in § 5):

THEOREM IV. *If $u = f(x_1, x_2, \cdots, x_n)$ is any solution of the wave equation* (1), *then*

$$u = \frac{1}{s^{n-2}} f(x_1/s^2, x_2/s^2, \cdots, x_n/s^2).$$

is also a solution.

THEOREM V. *If $u = f(x_1, x_2, \cdots, x_n)$ is any solution of equation* (17) *then*

$$u = f(x_1/s^2, x_2/s^2, \cdots, x_n/s^2).$$

is also a solution.

§ 10. *The case when σ is a linear function and $n > 2$.* In this case σ has the form given in equation (40). From (46) and the results at the end of § 4, the latter carried over to the inverse transformation, it follows that we have

$$1/\sigma = \gamma_1 y_1 + \gamma_2 y_2 + \cdots + \gamma_n y_n + \gamma. \tag{48}$$

* For the case $n = 3$ see Kelvin, *Liouville's Journal*, 12 (1847): 259. For general n see Bôcher, *Bull. Amer. Math. Soc.*, 9 (1903): 459. For $n = 4$ see Bateman, *Proc. Lond. Math. Soc.*, (2) 7 (1908): 75. The methods of the present paper are different from those of these authors.

where $\gamma_1, \gamma_2, \cdots, \gamma_n$ are constants such that

$$\gamma_1^2 + \gamma_2^2 + \cdots + \gamma_n^2 = 0. \tag{49}$$

From (40) and (48) it follows that

$$(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n + \beta)(\gamma_1 y_1 + \gamma_2 y_2 + \cdots + \gamma_n y_n + \gamma) = 1. \tag{50}$$

By aid of (23) equation (31) may be put in the form

$$\frac{\partial}{\partial x_j}\left(\sigma^2 \frac{\partial y_l}{\partial x_i}\right) = \sigma\left(\frac{\partial y_l}{\partial x_i}\frac{\partial \sigma}{\partial x_j} - \frac{\partial y_l}{\partial x_j}\frac{\partial \sigma}{\partial x_i} - \delta_{ij}\frac{\partial(1/\sigma)}{\partial y_l}\right).$$

In the present case, by aid of (40) and (48), this equation reduces to

$$\sigma \frac{\partial^2 y_l}{\partial x_i \partial x_j} + \beta_j \frac{\partial y_l}{\partial x_i} + \beta_i \frac{\partial y_l}{\partial x_j} + \delta_{ij}\gamma_l = 0.$$

On making the transformation

$$y_l = \sigma^{-1} t_l \tag{51}$$

the equation takes the form

$$\frac{\partial^2 t_l}{\partial x_i \partial x_j} + \delta_{ij}\gamma_l = 0.$$

On taking i and j different in this equation one sees that t_l has the form

$$t_l = T_{l1}(x_1) + T_{l2}(x_2) + \cdots + T_{ln}(x_n),$$

where $T_l(x_1)$ is a function of x_i alone. Moreover, on taking $j = i$ we see that each $T_{li}(x_i)$ has its second derivative with respect to x_i equal to $-\gamma_l$. Hence t_l can be written in the form

$$t_l = -\tfrac{1}{2}\gamma_l(x_1^2 + x_2^2 + \cdots + x_n^2) + \epsilon_{l1}x_1 + \epsilon_{l2}x_2 + \cdots + \epsilon_{ln}x_n + \eta_l,$$

where the ϵ's and η's are constants. Then from (51) it follows that we must have

$$\sigma y_i = -\tfrac{1}{2}\gamma_i(x_1^2 + x_2^2 + \cdots + x_n^2) + \epsilon_{i1}x_1 + \epsilon_{i2}x_2 + \cdots + \epsilon_{in}x_n + \eta_i, \quad i = 1, 2, \cdots, n, \tag{52}$$

where the constants β_i and γ_i satisfy conditions (41) and (49). Since it is

necessary and sufficient that the y_i shall satisfy equations (19), we now impose the conditions stated in these equations. Thus we have

$$\delta_{ij} = \sigma^2 \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k}$$

$$= \sum_{k=1}^{n} (\gamma_i x_k + \beta_k y_i - \epsilon_{ik})\ (\gamma_j x_k + \beta_k y_j - \epsilon_{jk})$$

$$= \sum_{k=1}^{n} \epsilon_{ik}\epsilon_{jk} + \gamma_i\eta_j + \gamma_j\eta_i - y_i\ [\sum_{k=1}^{n} \beta_k\epsilon_{jk} + \beta\gamma_j] - y_j\ [\sum_{k=1}^{n} \beta_k\epsilon_{ik} + \beta\gamma_i],$$

the last member being gotten by aid of equation (41). Hence, in order that the transformation defined in (52) shall be effective, it is necessary and sufficient that the constants involved shall satisfy the following conditions additional to that expressed in equation (41):

$$(53) \qquad \sum_{k=1}^{n} \epsilon_{ik}\epsilon_{jk} + \gamma_i\eta_j + \gamma_j\eta_i = \delta_{ij}, \quad \sum_{k=1}^{n} \beta_k\epsilon_{ik} + \beta\gamma_i = 0, \qquad i, j = 1, 2, \cdots, n.$$

It thus appears that condition (49) must be implied by (41) and (53), since there was no need of employing (49) in the derivation just given of the necessary and sufficient conditions that an effective transformation shall be afforded by equations (52).

We have seen that the γ's must satisfy (49). From (40), (50), (52) we see that the following conditions also must be satisfied:

$$(54) \qquad \sum_{k=1}^{n} \gamma_k\eta_k + \gamma\beta = 1, \quad \sum_{k=1}^{n} \gamma_k\epsilon_{ki} + \gamma\beta_i = 0, \quad i = 1, 2, \cdots, n.$$

It is clear, however, that these conditions on the constants are not independent of those previously derived.

Let us apply the linear transformations of § 6 to simplify transformation (52). By a translation of the x-space we may reduce β to zero. Similarly by a translation of the y-space we may reduce γ to zero. Now at least two of the quantities $\beta_1, \beta_2, \cdots, \beta_n$ are different from zero, since the sum of their squares is zero and we are supposing that σ is not a constant. Without loss of generality we suppose that $\beta_{n-1} \neq 0$, $\beta_n \neq 0$. If we now replace x_i by $\sqrt{-1}\ x_i'/\beta_n$, $i = 1, 2, \cdots, n$, we have a linear transformation belonging to those in § 6. In terms of these variables σ has the form

$$\sigma = \bar{\beta}_1 x_1' + \bar{\beta}_2 x_2' + \cdots + \bar{\beta}_{n-1} x'_{n-1} + \sqrt{-1}\ x'_n$$

where

$$\bar{\beta}_1{}^2 + \bar{\beta}_2{}^2 + \cdots + \bar{\beta}^2{}_{n-1} = 1.$$

Now if we revolve the $(n-1)$-space $x'_{n-1}=0$ in such a way as to leave the $(n-1)$-space $x_n'=0$ invariant, we can bring the former into coincidence with the $(n-1)$-space $\bar\beta_1 x_1' + \cdots + \bar\beta_{n-1}x'_{n-1}=0$. Let x_1'', x_2'', $\cdots$, x''_{n-1}, $x_n''=x_n'$ be the new coordinates after this revolution; then σ has the form $\sigma = x''_{n-1} + \sqrt{-1}\, x_n''$. The transformations employed all belong to the group of § 6. Hence, by means of transformations belonging to the linear group of § 6, we can reduce σ to a normal form in which the coefficients β are

$$\beta=\beta_1=\beta_2=\cdots=\beta_{n-2}=0, \quad \beta_{n-1}=1, \quad \beta_n=\sqrt{-1}.$$

Let us now take the original value of σ in this normal form. In view of the results in § 8 it is sufficient to obtain a single transformation of the type of (52). In order to find a convenient one we may also similarly normalize the y-coordinates so that $\gamma=\gamma_1=\gamma_2=\cdots=\gamma_{n-2}=0$, $\gamma_{n-1}=1$, $\gamma_n=\sqrt{-1}$. Then it is easy to obtain a particular solution of equations (53). Thus we have for a particular form of transformation (52) the following:

$$(55)\qquad \begin{cases} y_i = \dfrac{x_i}{x_{n-1}+\sqrt{-1}\, x_n}, & i=1, 2, \cdots, n-2, \\ y_{n-1} = \dfrac{1-s^2}{2(x_{n-1}+\sqrt{-1}\, x_n)}, & \\ y_n = \dfrac{1+s^2}{2\sqrt{-1}(x_{n-1}+\sqrt{-1}\, x_n)}, & \end{cases}$$

where $s^2 = x_1^2 + x_2^2 + \cdots + x_n^2$. For this transformation and the resulting theorems in the special case when $n=4$ see Bateman, *l. c.*, p. 77.

Then we have the following theorems:

THEOREM VI. *If $u=f(x_1, x_2, \cdots, x_n)$ is any solution of the wave equation* (1) *then*

$$u=(x_{n-1}+\sqrt{-1}\, x_n)^{\frac{1}{2}(2-n)} f(y_1, y_2, \cdots, y_n)$$

is also a solution, the functions y_i being defined by equations (55).

THEOREM VII. *If $u=f(x_1, x_2, \cdots, x_n)$ is any solution of equation* (17) *then $u=f(y_1, y_2, \cdots, y_n)$ is also a solution, the functions y_i being defined by equations* (55).

The transformation defined by equations (55) is an imaginary Cremona transformation of period two. A real Cremona transformation of period two in $2n$ variables may be obtained from it by separating the given variables into

real and imaginary parts and equating the real and imaginary parts of the two members of each of the resulting equations.

§ 11. *Summary of results.* Some results proved but not explicitly stated in the foregoing sections may be summarized in the form of the following theorems:

THEOREM VIII. *The transformation group involved in Theorem I and that involved in theorem III are the same. The group is that of the conformal transformations in space of n dimensions characterized at the end of* § 3, *imaginary transformations being allowed. The transformation group of theorem II is that subgroup of the foregoing in which* $\rho = 1$. (It is completely characterized in § 6.)

THEOREM IX. *The conformal transformation group of theorem I or III* (see end of § 3), *when* $n > 2$, *is generated by the linear transformations of* § 6 *and the two particular transformations* (47) *and* (55).

By means of the transformation indicated in the second paragraph of the introduction, it is seen that the two particular transformations (47) and (55) are real when they are taken in the form applicable to the wave equation as it is generally written by the mathematical physicists.

Transformations Leaving Invariant the Heat Equation of Physics.*

By John A. Goff.

We shall write the general heat equation in the form

$$\frac{\partial v}{\partial y_0} + \sum_{k=1}^{n} \frac{\partial^2 v}{\partial y_k^2} = 0. \tag{1}$$

For the case when $n = 3$, the transformation $y_0 = -c\rho t/k$, $y_1 = x$, $y_2 = y$, $y_3 = z$ reduces equation (1) to the classical form

$$c\rho \frac{\partial v}{\partial t} = k\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right).$$

of the heat equation of theoretical physics.

§ 1. *The Partial Differential Equations Defining the Transformation of the Heat Equation into Itself.* Let the function $v(y_0, y_1, y_2, \cdots, y_n)$ be a solution of the heat equation (1) and consider the transformation,

$$u = w \cdot v(y_0, y_1, \cdots, y_n), \qquad w \not\equiv 0, \tag{2}$$

where w is a funcion of $x_0, x_1, \cdots, x_n$. Also the functions $y_0, y_1, \cdots, y_n$ are to be functionally independent functions of $x_0, x_1, \cdots, x_n$ so determined, in connection with w, that whenever v satisfies equation (1), u will satisfy the following equation:

$$\frac{\partial u}{\partial x_0} + \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = 0. \tag{3}$$

We shall prove the following theorem:

Theorem I. *Let $w, y_0, y_1, \cdots, y_n$ be a set of functions of $x_0, x_1, \cdots, x_n$ such that $y_0, y_1, \cdots, y_n$ are functionally independent and $w \not\equiv 0$. Then a necessary and sufficient condition that every solution of* (1) *shall be transformed through* (2) *into a solution of* (3) *is that the functions $w, y_0, y_1, \cdots, y_n$ shall satisfy the following conditions: a function ρ exists, not identically zero, such that*

$$\frac{\partial w}{\partial x_0} + \sum_{k=1}^{n} \frac{\partial^2 w}{\partial x_k^2} = 0, \tag{4}$$

* Written with the aid of the foregoing paper by Professor Carmichael. References to Carmichael's paper are indicated by prefixing the letter C; thus, (C30) refers to Carmichael's equation (30).

$$(5)\qquad w\sum_{k=1}^{n}\frac{\partial^2 y_j}{\partial x_k^2}+2\sum_{k=1}^{n}\frac{\partial w}{\partial x_k}\frac{\partial y_j}{\partial x_k}+w\frac{\partial y_j}{\partial x_0}=0,\qquad j=1,2,\cdots,n,$$

$$(6)\qquad \sum_{k=1}^{n}\frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_k}=\delta_{ij}\rho,\qquad i,j=1,2,\cdots,n,$$

$$(7)\qquad w\sum_{k=1}^{n}\frac{\partial^2 y_0}{\partial x_k^2}+2\sum_{k=1}^{n}\frac{\partial w}{\partial x_k}\frac{\partial y_0}{\partial x_k}+w\frac{\partial y_0}{\partial x_0}=w\rho,$$

$$(8)\qquad \sum_{k=1}^{n}\frac{\partial y_0}{\partial x_k}\frac{\partial y_j}{\partial x_k}=0,\qquad j=0,1,2,\cdots,n.$$

Since 1, y_j, y_iy_j $(i\neq j)$, and $y_j^2-y_n^2$ $(i,j=1,2,\cdots,n)$ are solutions of (1), it follows that the functions w and y_j must be such that w, wy_j, wy_iy_j $(i\neq j)$, and $w(y_j^2-y_n^2)$ for $i,j=1,2,\cdots,n$ are solutions of (3). These facts imply equations (4), (5), and (6). Substituting the value of u from (2) into (3) and simplifying by aid of (1), (4), (5), (6), we have

$$\frac{\partial v}{\partial y_0}\left\{w\sum_{k=1}^{n}\frac{\partial^2 y_0}{\partial x_k^2}+2\sum_{k=1}^{n}\frac{\partial w}{\partial x_k}\frac{\partial y_0}{\partial x_k}+w\frac{\partial y_0}{\partial x_0}-w\rho\right\}$$
$$+2w\sum_{j=1}^{n}\frac{\partial^2 v}{\partial y_0\partial y_j}\sum_{k=1}^{n}\frac{\partial y_0}{\partial x_k}\frac{\partial y_j}{\partial x_k}$$
$$+w\frac{\partial^2 v}{\partial y_0^2}\sum_{k=1}^{n}\left(\frac{\partial y_0}{\partial x_k}\right)^2=0.$$

The particular solutions of (1), namely, $y_0-\frac{1}{2}y_j^2$ and $y_0y_j-\frac{1}{6}y_j^3$ $(j=1,2,\cdots,n)$, show that (7) must be satisfied and that (8) must be satisfied for $j=1,2,\cdots,n$. That (8) must also be satisfied for $j=0$ then follows immediately. The conditions of the theorem, with the exception of the condition that ρ be not identically zero, are thus shown to be necessary.

Formulae (C19) and (C21) can be used since both of these are implied by (6). Let us multiply (8) by $\partial y_j/\partial y_l$, sum as to j from 1 to n, and simplify by means of (C21) getting

$$(9)\qquad \frac{\partial y_0}{\partial x_l}=0,\qquad l=1,2,\cdots,n.$$

Then (7) becomes

$$(10)\qquad \frac{\partial y_0}{\partial x_0}=\rho.$$

The Jacobian (functional determinant) is

$$J=\left|\frac{\partial y_j}{\partial x_k}\right|,\qquad j,k=0,1,2,\cdots,n,$$

which becomes, by virtue of (9) and (10),

$$J = \rho \left| \frac{\partial y_j}{\partial x_k} \right| \qquad j, k = 1, 2, \cdots, n,$$

so that

$$J = \rho^{\frac{n+2}{2}} \qquad \text{by (C20)} \tag{11}$$

Consequently it is necessary and sufficient that ρ be not identically zero in order that the functions $y_0, y_1, \cdots, y_n$ be functionally independent functions of $x_0, x_1, \cdots, x_n$. Moreover conditions (4) to (8) can easily be seen to be sufficient conditions. Hence the theorem.

§ 2. *Determination of* $w, y_0, y_1, \cdots, y_n$. From equations (9) and (10) we get

$$\frac{\partial \rho}{\partial x_l} = 0, \qquad l = 1, 2, \cdots, n, \tag{12}$$

so that ρ is a function of x_0 alone. If we consider the equations

$$x_i = x_i(y_0, y_1, \cdots, y_n), \qquad i = 1, 2, \cdots, n,$$

of the inverse transformation, then on differentiating with respect to x_j we get

$$\delta_{ij} = \sum_{k=0}^{n} \frac{\partial x_i}{\partial y_k} \frac{\partial y_k}{\partial x_j}, \qquad i, j = 1, 2, \cdots, n.$$

Then by virtue of (9) we have

$$\delta_{ij} = \sum_{k=1}^{n} \frac{\partial x_i}{\partial y_k} \frac{\partial y_k}{\partial x_j}, \qquad i, j = 1, 2, \cdots, n.$$

Hence we can use equations (C31) and (C23) together with (12) to show that

$$\frac{\partial^2 y_j}{\partial x_k \partial x_l} = 0, \qquad j, k, l = 1, 2, \cdots, n. \tag{13}$$

The functions y_j are then of the form

$$y_j = a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n + a_j, \qquad j = 1, 2, \cdots, n. \tag{14}$$

where the a's are functions of x_0 alone.

In view of (13), equation (5) becomes

$$\frac{\partial y_j}{\partial x_0} = -2 \sum_{m=1}^{n} \frac{\partial (\log w)}{\partial x_m} \frac{\partial y_j}{\partial x_m}, \qquad j = 1, 2, \cdots, n. \tag{15}$$

Now (6) implies that

$$\sum_{j=1}^{n} \frac{\partial y_j}{\partial x_k} \frac{\partial y_j}{\partial x_l} = \delta_{kl}\rho, \qquad k, l = 1, 2, \cdots, n. \tag{C21}$$

Differentiate both sides of this equation with respect to x_0 and then by means of (15) obtain

$$\sum_{m=1}^{n} \frac{\partial^2(\log w)}{\partial x_m \partial x_l} \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_k} \frac{\partial y_j}{\partial x_m} + \sum_{m=1}^{n} \frac{\partial^2(\log w)}{\partial x_m \partial x_k} \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_l} \frac{\partial y_j}{\partial x_m}$$

$$+ \sum_{m=1}^{n} \frac{\partial(\log w)}{\partial x_m} \frac{\partial}{\partial x_m} \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_k} \frac{\partial y_j}{\partial x_l} = -\tfrac{1}{2} \delta_{kl} \frac{\partial \rho}{\partial x_0}, \qquad k, l = 1, 2, \cdots, n.$$

When this is simplified by means of (C21) and (12), the result is

$$(16) \qquad \frac{\partial^2(\log w)}{\partial x_k \partial x_l} = -\tfrac{1}{4} \delta_{kl} \rho'/\rho = -\tfrac{1}{4} \delta_{kl} \zeta, \qquad k, l = 1, 2, \cdots, n.$$

where ρ' denotes the derivative of $\rho(x_0)$ with respect to its argument. For convenience we have denoted the ratio ρ'/ρ by ζ.

From (15) with the aid of (13), (14) and (16), we get

$$\frac{\partial^2 y_j}{\partial x_0 \partial x_l} = \tfrac{1}{2} \zeta \cdot a_{jl}, \qquad j, l = 1, 2, \cdots, n.$$

But from (14) alone we can obtain directly,

$$\frac{\partial^2 y_j}{\partial x_0 \partial x_l} = a'_{jl}, \qquad j, l = 1, 2, \cdots, n.$$

where the prime denotes the derivative with respect to x_0. Consequently $a'_{jl} = \frac{1}{2} a_{jl} \rho'/\rho$, whence

$$(17) \qquad a_{jl} = C_{jl} \rho^{\frac{1}{2}}, \qquad j, l = 1, 2, \cdots, n,$$

where the quantities C_{jl} are constants. These results show that the y_j can be written in the form,

$$(18) \qquad y_j = \rho^{\frac{1}{2}} \{ \sum_{m=1}^{n} C_{jm} x_m + \gamma_j \}$$

where, by virtue of (6) and (C21) the constants satisfy the conditions

$$(19) \qquad \sum_{k=1}^{n} C_{jk} C_{ik} = \delta_{ij}, \qquad i, j = 1, 2, \cdots, n,$$

and the γ_j are functions of x_0 alone.

From (16) we see that $\log w$ is of the form

$$(20) \qquad \log w = -\tfrac{1}{8} \zeta \sum_{m=1}^{n} (x_m - \beta_m)^2 + \beta,$$

where the functions β are functions of x_0 alone. Moreover, since w must satisfy (4), $\log w$ must satisfy the relation

$$(21) \qquad \frac{\partial(\log w)}{\partial x_0} + \sum_{k=1}^{n} \frac{\partial^2(\log w)}{\partial x_k^2} + \sum_{k=1}^{n} \{ \frac{\partial(\log w)}{\partial x_k} \}^2 = 0.$$

From this, with the aid of (20), we obtain the following equation:

$$(22)\qquad \tfrac{1}{4}\zeta \sum_{m=1}^{n} (x_m - \beta_m)\beta'_m + \beta' - (n/4)\,\zeta + (1/16\,\zeta^2 - \tfrac{1}{8}\zeta') \sum_{m=1}^{n} (x_m - \beta_m)^2 = 0.$$

This equation implies that

$$(23)\qquad \zeta^2 = 2\zeta' \qquad \text{and} \qquad \beta' = (n/4)\,\zeta;$$

and, if $\zeta \not\equiv 0$, that

$$(24)\qquad \beta'_m = 0, \qquad m = 1, 2, \cdots, n.$$

Now form $\partial x_j/\partial x_0$ from equation (18) and compare with $\partial y_j/\partial x_0$ obtained from (15) with the aid of (18) and (20). This leads to the following differential equation:

$$(25)\qquad \gamma'_j + \tfrac{1}{2}\zeta\gamma_j + \tfrac{1}{2}\zeta \sum_{m=1}^{n} C_{jm}\beta_m = 0, \qquad j = 1, 2, \cdots, n.$$

Remembering that $\zeta = \rho'/\rho$ and supposing for the moment that ρ is not a constant, we then see, by aid of (24), that

$$(26)\qquad \gamma_j = -\sum_{m=1}^{n} C_{jm}\beta_m + C_j/\rho^{\frac{1}{2}}, \qquad j = 1, 2, \cdots, n$$

where the C_j are constants. In this case the β_m are also constants. Consequently (18) becomes

$$(27)\qquad y_j = \rho^{\frac{1}{2}} \sum_{m=1}^{n} C_{jm}(x_m - \beta_m) + C_j, \qquad j = 1, 2, \cdots, n,$$

when ρ is not a constant.

§ 3. *The Case when* $\zeta \equiv 0$. In this case $\rho = k^2 =$ constant. From (25) we see that each γ_j is a constant which we may choose in such a way as to enable us to write

$$(28)\qquad y_j = k \sum_{m=1}^{n} C_{jm}(x_m - \beta_m) \qquad j = 1, 2, \cdots, n.$$

where the constants C_{jm} must satisfy conditions (19); namely,

$$(19)\qquad \sum_{m=1}^{n} C_{jm}C_{im} = \delta_{ij}, \qquad j = 1, 2, \cdots, n.$$

From equation (10) we see that y_0 is of the form

$$(29)\qquad y_0 = k^2(x_0 - \beta_0).$$

Furthermore from (2) and the last of (23) (which becomes $\beta' = 0$) we see that w is a constant and without loss of generality we may take

$$(30)\qquad w = 1.$$

If $\rho^{\frac{1}{2}} = 1$, we may write (28) and (29) as follows

$$y_j = \sum_{m=0}^{n} C_{jm}(x_m - \beta_m), \qquad j = 0, 1, 2, \cdots, n,$$

where $C_{0m} = C_{m0} = \delta_{0m}$. Then

$$\sum_{k=0}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} = \delta_{ij}, \qquad i, j = 0, 1, 2, \cdots, n;$$

for if $j = 0$ and $i \neq 0$, it reduces to 0; if $j = 0$ and $i = 0$, it reduces to 1; while if $i \neq 0$ and $j \neq 0$ it becomes simply δ_{ij} by (6). These results show that the transformation described by equations (28), (29), (19), and (30) is conformal when $\rho^{\frac{1}{2}} = 1$.

§ 4. *The Case when* $\zeta \neq 0$. By using equations (23) we get the following particular solutions:

$$(31) \qquad \zeta = -2/x_0, \quad \rho = 1/x_0^2, \quad \beta = \log x_0^{-(n/2)};$$

whence by aid of (20) and (27) we have

$$(32) \qquad w = x_0^{-(n/2)} \exp\left\{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{4x_0}\right\}; \quad y_j = x_j/x_0, \quad j = 1, 2, \cdots, n.$$

By (10) we have

$$(33) \qquad y_0 = -1/x_0.$$

The constants which have been omitted in obtaining these particular solutions can be removed by a transformation described under § 3. Thus we can state the following theorem:*

THEOREM II. *If* $u = f(x_0, x_1, \cdots, x_n)$ *is any solution of the heat equation* (3), *then*

$$u = x_0^{-(n/2)} \exp\left\{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{4x_0}\right\} f(-1/x_0, x_1/x_0, x_2/x_0, \cdots, x_n/x)$$

is also a solution of (3).

§ 5. *Summary. The results show that the transformation of theorem II together with the transformations described by equations* (28), (29), (19), *and* (30) *generate all the transformations of the form of equation* (2) *which leave the heat equation invariant.*

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* For the case $n = 3$ see J. Brill, *Messenger of Mathematics*, 2 (1891), p. 137; for the case $n = 1$ see P. Appell, *Journal de Mathématiques*, (4) 8 (1892), p. 187.

The Application of Fractional Operators to Functional Equations.*

By H. T. Davis.

1. *Fractional Operations.* In a recent paper † the author showed how it was possible to express certain types of integral equations as differential equations of fractional order and by operating on the latter to obtain solutions of the original equations. Since that time several examples of such equations have come to his attention, but the methods of solution employed in these cases made no use of the idea of fractional operations. For example, John R. Carson in a series of papers ‡ has made an analysis of the calculus which Oliver Heaviside developed and used so effectively in the circuit problems of electrical theory.§ Carson shows very elegantly how the Heaviside calculus is formally equivalent to the process of solving an integral equation of the form

$$f(x) = \int_0^\infty e^{-xt} u(t)\, dt,$$

but he does not point out the interesting fact that the most striking feature of the Heaviside calculus is also a special application of the fractional operators of Riemann.‖ In particular, the series expansion of the Heaviside operator for the general case of the non-inductive cable which Carson characterizes as "hopelessly complicated to either interpret or compute" is reduced with considerable simplicity by the methods of this paper.¶

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* Presented to the Indiana Section of the Mathematical Association of America, May 8, 1926.

† "Fractional Operations as Applied to a Class of Volterra Integral Equations," *American Journal of Mathematics*, Vol. 46 (1924), pp. 95-109.

‡ "Theory of the Transient Oscillations of Electrical Networks and Transmission Systems," *Trans. A. I. E. E.* (1919); "The Heaviside Operational Calculus," *The Bell System Tech. Journal*, (1922); "Electric Circuit Theory and the Operational Calculus," *Ibid.*, Vol. 4 (1925), pp. 685-761; Vol. 5 (1926), pp. 50-95; *Bull. Amer. Math. Soc.*, Vol. 32 (1926), pp. 43-63.

§ *Electromagnetic Theory*, Vol. II.

‖ A similar observation has recently been made by P. Levy: "Le calcul symbolique d'Heaviside," *Bulletin des sciences Mathématiques*, Vol. 50, 2nd series (June, 1926), pp. 174-192.

¶ *Bell System Tech. Journal*, Vol. 5 (1926), p. 59.

It seems important, therefore, that some effort should be made to give wider introduction to these fractional operators and to suggest that a standard symbolism be adopted for them. The great elegance that can be secured by proper use of fractional operators and the power which they have in the solution of complicated functional equations should more than justify a more general recognition and use. The aim of the present paper is to present the case of the "fractional calculus."

The idea of fractional differentiation was clearly in the mind of Leibnitz who said in a letter to G. F. A. de l'Hospital in 1695 that he was considering the meaning to be attached to $d^{\frac{1}{2}}/dx^{\frac{1}{2}}$. The first real use of fractional operations was apparently made by Abel who, as early as 1823 used both the symbols $d^{-\frac{1}{2}}/dx^{-\frac{1}{2}}\ \psi(x) = 1/\sqrt{\pi}\ \int^{\frac{1}{2}} \psi(x)\ dx$ and $\psi(x) = \sqrt{\pi}\ d^{\frac{1}{2}}s/dx^{\frac{1}{2}}$ in solving the problem of the tautochrone.*

Although there is mention of fractional operators in the works of Euler, Laplace and Fourier, it was J. Liouville who first made an extensive study of the new calculus. Altogether he devoted eight memoirs totaling about three hundred pages to the subject making application to problems in geometry and mechanics.† The effectiveness of this work was somewhat limited, however, by the fact that a specialized definition of fractional operations was employed. Most of the modern work in this field goes back to Riemann who, in a paper developed during his student days but posthumously published, gave general definitions of fractional differentiation and integration.‡

Slightly generalized these definitions may be stated as follows:

By the *fractional integration* of a function $u(x)$ we shall mean the operation

$$ {}_cD_x^{-\nu}u(x) = \int_c^x \frac{(x-t)^{\nu-1}}{\Gamma(\nu)}\ u(t)\ dt, \qquad \nu > 0, \tag{1} $$

and by *fractional differentiation,*

$$ {}_cD_x^{m+\nu}u(x) = \frac{d^{m+1}}{dx^{m+1}} \int_c^x \frac{(x-t)^{-\nu}}{\Gamma(1-\nu)}\ u(t)\ dt, \quad m = 0,\ 1,\ 2,\ \cdots \quad 0 < \nu < 1. \tag{2} $$

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* "Solution de quelques problèmes a l'aide d'integrales définies." *Werke* (1881), Vol. 1, pp. 10-27.

† *Journal de l'école polytechnique,* cah. 21 (1832), pp. 1-69; 71-162; 163-186; cah. 24 (1835), pp. 17-54; cah. 25 (1836), pp. 56-84; *Journal für Mathematik,* Vol. 11 (1834), pp. 1-19; Vol. 12 (1834), pp. 273-287; Vol. 13 (1835), pp. 219-232.

‡ "Versuch einer allgemeinen Auffassung der Integration und Differentiation." *Gesammelte Werke,* Leipzig (1892), pp. 353-366.

It is important to note that the constant "c" is a vital part of the operational symbol. Thus Riemann let $c = 0$ in his original definition and Liouville let $c = -\infty$.

Various symbols have been suggested for these operators. We have already mentioned d^ν/dx^ν and $d^{-\nu}/dx^{-\nu}$, but they are somewhat cumbersome and do not convey information regarding the lower limit of the integral. This symbolism was effectively used by Liouville who, as has been stated, employed a fixed limit. Emil Post in discussing the solution of the problem $d^2y/dx^2 - y/x = 0$ in the *Mathematical Monthly* several years ago * suggested the use of the symbol $\{d^{-\nu}/dx^{-\nu}\}_c^x$. Riemann used $\partial_x^{-\nu}$ and ∂_x^ν for his operators and P. Levy in a recent paper † employed the symbols $I_x^\nu[\phi(x)]$ for integration and $D_x^\nu[\phi(x)]$ for differentiation where the lower limit of the integral was zero. In a contribution to the subject by H. Laurent in 1883 we find the operation expressed by $\left|\frac{d^\nu f}{dx^\nu}\right|_c^x$.‡ J. Hadamard, who made fundamental use of Riemann's operator and gave some generalization to it in his work on analytic functions, employed $D_x^\nu\phi(x)$.§ The mathematical encyclopedia, to which the reader is referred for a longer bibliography on the subject, uses $D_c^{x\nu} f(x)$ as a symbol for the general operator.‖ Since all of the information is conveyed in a simple manner by the symbols ${}_cD_x^{-\nu}u(x)$ and ${}_cD_x^\nu u(x)$ and since they are only slight variations of those which appear to have had the largest usage, the author has adopted them in his work. It is occasionally convenient to use the easily understood abbreviations $u^{(-\nu)}(x)$ and $u^{(\nu)}(x)$ when there is no question as to the lower limit of integration.

2. *Some Properties of the Operators.* For convenience in reference a few of the properties of fractional operators will be stated. Where proofs have been omitted they will be found elsewhere.¶

The significance of the definitions given in the preceding section attaches to the fact that

$$\begin{aligned} {}_cD_x^\mu[u(x) + v(x)] &= {}_cD_x^\mu u(x) + {}_cD_x^\mu v(x), \\ {}_cD_x^\mu k\, u(x) &= k\, {}_cD_x^\mu u(x), \\ {}_cD_x^\mu\, {}_cD_x^\nu\, u(x) &= {}_cD_x^{(\mu+\nu)}u(x). \end{aligned} \tag{3}$$

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* *Mathematical Monthly*, Vol. 26 (1919), p. 37; see also Vol. 25 (1918), p. 172, problem 433.

† *Bulletin des Sciences Mathématiques*, Vol. 47 (1923), pp. 307 and 343.

‡ *Nouvelles Annales*, series 3, Vol. 3 (1883), p. 240.

§ *Journal de mathématiques*, Vol. 57 (1892), p. 154.

‖ *Encyklopädie der Mathematischen Wissenschaften*, II, A, 1, pp. 116-119; II, A, 11, pp. 770-772. Also *Encyclopédie des Sciences Mathématiques*, T. 11, V. 5, fas. 1, 8.

¶ Davis, *loc. cit.*

The analogue of Taylor's theorem is found in the following development of the integration operator:

$$ {}_cD_x^{-\nu}\, u(x) = \frac{(x-c)^\nu}{\Gamma(\nu+1)} \sum_{m=0}^{\infty} (-1)^m \frac{\nu}{\nu+m} \frac{(x-c)^m}{m!}\, u^{(m)}(x). \quad (4) $$

This is easily proved by replacing $u(t)$ in (1) by its expansion about the point x. We thus get,

$$ {}_cD_x^{-\nu}\, u(x) = \int_c^x \frac{(x-t)^{\nu-1}}{\Gamma(\nu)} \sum_{m=0}^{\infty} u^{(m)}(x) \frac{(t-x)^m}{m!}\, dt, $$

which easily reduces to (4).

Another useful expansion is that analogous to the formula of Leibnitz which may be stated as follows:

$$ {}_cD^{x-\nu}\,[U(x)\, V(x)] = U(x)\, V^{(-\nu)}(x) - \nu U'(x)\, V^{(-\nu-1)}(x) + \frac{\nu(\nu+1)}{2!}\, U''(x)\, V^{(-\nu-2)}(x) - \cdots. \quad (5) $$

In the application of these formulas it is often useful to note the following derivative:

If $v(x) = \int_c^x \frac{g(x,s)\, u(s)}{(x-s)^\nu}\, ds$, we then have

$$ v'(x) = \frac{g(x,c)\, u(c)}{(x-c)^\nu} + \int_c^x \frac{D_x g(x,s) + D_s g(x,s)}{(x-s)^\nu}\, u(s)\, ds + \int_c^x \frac{g(x,s)}{(x-s)^\nu}\, u'(s)\, ds. \quad (6) $$

This formula can be established by the method of "partie finie" due simultaneously to R. D'Adhemar and J. Hadamard in 1904.*

Let us write

$$ v_\epsilon(x) = \int_c^{x-\epsilon} \frac{g(x,s)\, u(s)}{(x-s)^\nu}\, ds; $$

since there is no longer a singularity in the integrand for the upper limit of the integral, we can apply the ordinary rule for the differentiation of a function defined by an integral. We then have

$$ \frac{dv_\epsilon(x)}{dx} = \int_c^{x-\epsilon} \left\{ \frac{\partial/\partial x\; g(x,s)\, u(s)}{(x-s)^\nu} + g(x,s)\, u(s)\; \partial/\partial x\, (x-s)^{-\nu} \right\} ds + \frac{g(x, x-\epsilon)\, u(x-\epsilon)}{\epsilon^\nu}. $$

* R. D'Adhemar, Thesis, *Journal de Math.* (1904); see also *Exercices et leçons d'analyse*, Paris (1908), p. 150; J. Hadamard, *Congres de Mathém.* (1904), pp. 265-271.

Since $\partial/\partial x\,(x-s)^{-\nu} = -\partial/\partial s\,(x-s)^{-\nu}$, we can substitute this value in the integrand of the preceding integral and integrate by parts. We then get,

$$\frac{dv_\epsilon(x)}{dx} = \int_0^{x-\epsilon} \frac{\partial/\partial x\; g(x,s)\,u(s) + \partial/\partial s\,[g(x,s)\,u(s)]}{(x-s)^\nu}\; ds + \frac{g(x,c)\,u(c)}{(x-c)^\nu}.$$

and in the limit we have formula (6).

It will be useful in another section of the paper to know that the well-known operator: *

$$(d/dx + k)^n f(x) = e^{-kx}\; d^n/dx^n\,\{e^{kx} f(x)\}, \quad n \text{ a positive integer},$$

is also valid for fractional values of n. This can be proved by means of formula (5).

We first write,

$$e^{-kx}\; d^n/dx^n\,\{e^{kx} f(x)\} = e^{-kx}\; \frac{d^{m-\mu}}{dx^{m-\mu}}\;\{e^{kx} f(x)\},$$

where m is an integer and μ a fraction between 0 and 1. Then in (5), setting $U(x)$ equal to e^{kx} and $V(x)$ equal to $f(x)$, we get,

$$e^{-kx}\; \frac{d^{m-\mu}}{dx^{m-\mu}}\;\{e^{kx} f(x)\} =$$

$$e^{-kx}\; d^m/dx^m \left[e^{kx}\,\{f^{(-\mu)}(x) - \mu k f^{(-\mu-1)} + \frac{\mu(\mu+1)}{2!}\; k^2 f^{(-\mu-2)} + \cdots\}\right] =$$

$$e^{-kx}\; d^m/dx^m\,[e^{kx}\,\{1 - \mu k z^{-1} + \frac{\mu(\mu+1)}{2!}\; k^2 z^{-2} + \cdots\} f^{(-\mu)}(x)],$$

where we have used the abbreviation $z = d^{-1}/dx^{-1}$. This expression then reduces to

$$e^{-kx}\; d^m/dx^m\,[e^{kx}\,(1 + k/z)^{-\mu} f^{(-\mu)}(x)] =$$
$$(d/dx + k)^m\,(1 + k/z)^{-\mu} f^{(-\mu)}(x) = (d/dx + k)^{m-\mu} f(x).$$

3. *The Solution of Fractional Equations.* One of the simplest cases of fractional equations is that of the linear equation with constant coefficients involving derivatives (or integrals) of orders which differ by multiples of $1/n$, n an integer, from one another. Thus let us consider the equation,

$$L(u) \equiv A_0 u + A_1 u^{(1/n)} + A_2 u^{(2/n)} + \cdots + A_{n-1} u^{(1-1/n)} = f(x), \quad (7)$$

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* J. Edwards, *Differential Calculus*, 3rd ed., London (1904), p. 70.

where we have designated by $u^{(v)}$ the derivative ${}_cD_x^v u(x)$. The function $f(x)$ will be characterized later.

If we replace $u(x)$ by $d/dx \int_c^x u(s)\, ds$ and the fractional derivatives by their formal values we see that (7) is equivalent to the following functional equation:

$$\frac{d}{dx}\int_c^x \frac{G(x,s)\, u(s)}{(x-s)^{1-1/n}}\, ds = f(x),$$

where

$$G(x,s) \equiv A_0(x-s)^{1-1/n} + \frac{A_1}{\Gamma(1-1/n)}(x-s)^{1-2/n} + \cdots + \frac{A_{n-1}}{\Gamma(1/n)},$$

Upon integrating both sides we arrive at the integral equation:

$$\int_c^x \frac{G(x,s)}{(x-s)^{1-1/n}}\, u(s)\, ds = \int_c^x f(s)\, ds.$$

The existence theorem is well known and has been stated under a variety of conditions for $f(x)$.* It is sufficient to assume that $f(x)$ is analytic in the neighborhood of $x = c$. Since the first equation is derived from the latter by differentiation, it is clear that a solution of the integral equation will also be a solution of its derivative.

Having thus assured ourselves of the existence of at least one solution of equation (7) we can proceed to find this solution by the following formal method. This method will be found to apply equally well to an equation involving fractional integrals differing by multiples of $1/n$ from one another and since the details are essentially the same no separate discussion will be made of this case.

Representing (7) symbolically by P,

$$P \equiv A_0 + A_1 z^{1/n} + A_2 z^{2/n} + \cdots + A_{n-1} z^{1-1/n},$$

we can rationalize this algebraic equation in the following manner. Multiplying P successively by $z^{1/n}$, $z^{2/n}$, etc. we get the system of equations,

$$\begin{aligned}
P &= A_0 + A_1 z^{1/n} + A_2 z^{2/n} + \cdots + A_{n-1} z^{1-1/n},\\
z^{1/n}P &= zA_{n-1} + A_0 z^{1/n} + A_1 z^{2/n} + \cdots + A_{n-2} z^{1-1/n},\\
z^{2/n}P &= zA_{n-2} + zA_{n-1} z^{1/n} + A_0 z^{2/n} + \cdots + A_{n-3} z^{1-1/n},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
z^{1-1/n}P &= zA_1 + zA_2 z^{1/n} + zA_3 z^{2/n} + \cdots + A_0 z^{1-1/n}.
\end{aligned}$$

* T. Lalesco, *Journal de Mathématique*, Vol. 73 (1908), pp. 125-202.

Eliminating $z^{1/n}, z^{2/n}, \cdots$ from the right hand side of these equations we obtain

$$\begin{vmatrix} A_0 - P, & A_1, & A_2, & \cdots, & A_{n-1} \\ zA_{n-1} - z^{1/n}P, & A_0, & A_1, & \cdots, & A_{n-2} \\ zA_{n-2} - z^{2/n}P, & zA_{n-1}, & A_0, & \cdots, & A_{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ zA_1 - z^{1-1/n}P, & zA_2, & zA_3, & \cdots, & A_0 \end{vmatrix} = 0.$$

From this it follows that

$$P\Delta(z) = A(z),$$

where

$$\Delta(z) = \begin{vmatrix} 1, & A_1, & A_2, & \cdots, & A_{n-1} \\ z^{1/n}, & A_0, & A_1, & \cdots, & A_{n-2} \\ z^{2/n}, & zA_{n-1}, & A_0, & \cdots, & A_{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z^{1-1/n}, & zA_2, & zA_3, & \cdots, & A_0 \end{vmatrix} \tag{8}$$

and $A(z)$ is the same determinant with the first column replaced by $A_0, zA_{n-1}, zA_{n-2}, \cdots, zA_1$.

Since $A(z)$ is a polynomial in z of degree less than or equal to $n-1$, it is clear that Δ is a rationalizing factor and equation (7) can be formally replaced by the ordinary linear equation

$$A(d/dx)\, u = \Delta(d/dx)\, f(x). \tag{9}$$

To this equation there corresponds an integral of the form

$$u(x) = \int_c^x W(x,t)\, \Delta(d/dt)\, f(t)\, dt, \tag{10}$$

where $W(x,t)$ is the Cauchy function familiar from the method of variation of parameters, provided $W(x,t)\, \Delta(d/dt)\, f(t)$ does not become infinite to a higher order than $\mu < 1$ at $t = c$.

To show that (10) is the unique solution of (7) we shall need the following theorem:

THEOREM 1. *The homogeneous equation* $L(u) = 0$ has no solution *except the trivial one* $u \equiv 0$, *provided* $c \neq \infty$.

Proof: Equation (7) is equivalent to the equation

$$A_0u(x) + d/dx \int_c^x \left\{ \frac{A_1(x-s)^{-(1/n)}}{\Gamma(1-1/n)} + \frac{A_2(x-s)^{-(2/n)}}{\Gamma(1-2/n)} + \cdots + \frac{A_{n-1}(x-s)^{-(1-1/n)}}{\Gamma(1/n)} \right\} u(s)\, ds = 0, \tag{11}$$

which may be written from (6) in the form

$$L(u) \equiv A_0u(x) + \int_c^x I(x,s)\, u'(s)\, ds + u(c)\, I(x,c) = 0,$$

where $I(x,s)$ is the integrand of (11). Now unless $u(c) = 0$ we should have a contradiction since $\lim_{x=c} L(u)$ would then be infinity instead of zero. Similarly we may prove that $u(c) = u'(c) = \cdots = u^{(n)}(c) = 0$, which would not be the case unless c were a singular point of the equation $A(d/dx) = 0$. But since $c = \infty$ is the only singular point we conclude that $L(u) = 0$ can have no solution except the trivial one.

As illustration we shall consider two special equations which are of frequent occurrence in application.

I. As a special case of (7) we have

$${}_cD^{\frac{1}{2}}_x\, u + \lambda u = f(x). \tag{12}$$

Rationalizing we obtain the ordinary equation

$$d/dx\, u - \lambda^2 u = {}_cD_x^{\frac{1}{2}} f(x) - \lambda f(x), \tag{13}$$

whose solution is

$$u(x) = e^{\lambda^2 x} \int_c^x \{ ({}_cD_t^{\frac{1}{2}} - \lambda)\, f(t) \} e^{-\lambda^2 t}\, dt. \tag{14}$$

In order to see that this solution behaves properly at the point $x = c$, write (12) in the expanded form

$$\frac{u(c)}{\sqrt{\pi}\,(x-c)^{\frac{1}{2}}} + 1/\sqrt{\pi} \int_c^x \frac{u'(t)}{(x-t)^{\frac{1}{2}}}\, dt + \lambda u(x) = f(x).$$

If $f(x)$ is bounded at the point $x = c$, then it is clear that $u(c)$ must be zero if we are to avoid a contradiction. Since (14) is the only solution of (13) which satisfies this condition it must be the solution of (12).

Example: Solve ${}_0D_x^{\frac{1}{2}}\, u - u = -1$.

We get as the formal equivalent of this equation the ordinary equation

$$u' - u = -(1 + 1/\sqrt{\pi x}),$$

whose solution is

$$u(x) = e^x \int_0^x e^{-t} \left(-1 - 1/\sqrt{\pi t}\right) dt,$$

$$= 1 - e^x - 1/\sqrt{\pi x} \left\{(2x) + \frac{(2x)^2}{1 \cdot 3} + \frac{(2x)^3}{1 \cdot 3 \cdot 5} + \cdots\right\}.$$

This function can be shown by direct substitution to satisfy the original equation.

II. Consider also

$$ {}_cD_x^{-\frac{1}{2}} u + \lambda u = f(x), \tag{15}$$

whose rational equivalent is the equation

$$\lambda^2 \, d/dx \; u - u = \lambda f'(x) - {}_cD_x^{\frac{1}{2}} f(x). \tag{16}$$

The solution of this equation is

$$u(x) = C\, e^{1/\lambda^2 x} + e^{1/\lambda^2 x} \int_c^x \{1/\lambda f'(t) - 1/\lambda^2 \, {}_cD_t^{\frac{1}{2}} f(t)\}\, e^{-1/\lambda^2 t}\, dt. \tag{17}$$

In order to determine C consider equation (15) in the form

$$1/\sqrt{\pi} \int_c^x \frac{u(t)}{(x-t)^{\frac{1}{2}}}\, dt + \lambda u(x) = f(x).$$

It is clear that for $x = c$ we have $\lambda u(c) = f(c)$. Then from (17) we get $u(c) = C\, e^{1/\lambda^2 c} = 1/\lambda\, f(c)$ and (17) can be written in the form

$$u(x) = 1/\lambda\, f(c)\, e^{1/\lambda^2 (x-c)} + \int_c^x \{1/\lambda\, f'(t) - 1/\lambda^2 \, {}_cD_t\, f(t)\}\, e^{1/\lambda^2 (x-t)}\, dt \tag{18}$$

Example: Solve

$$ {}_0D_x^{-\frac{1}{2}} u + u = 1.$$

This rationalizes to the equation

$$u' - u = -\frac{1}{\sqrt{\pi x}},$$

whose solution is

$$u(x) = e^x - e^x \int_0^x e^{-t}\, 1/\sqrt{\pi t}\, dt.$$

This solution can be expanded in two ways:

$$u_1 = e^x - \frac{1}{\sqrt{\pi x}} \left\{(2x) + \frac{(2x)^2}{1 \cdot 3} + \frac{(2x)^3}{1 \cdot 3 \cdot 5} + \cdots\right\},$$

and $$u_2 = \frac{1}{\sqrt{\pi x}} \left\{1 - \frac{1}{2x} + \frac{1 \cdot 3}{(2x)^2} - \frac{1 \cdot 3 \cdot 5}{(2x)^3} + \cdots\right\}.$$

The second, although everywhere divergent, is the asymptotic expansion of the first.

4. *Generalization of Abel's Integral Equation.* Among special types of integral equations we find the following:

$$k x^a = \int_0^x \sqrt{x-t}\, u(t)\, dt, \tag{19}$$

which was first solved by N. Hirakawa * and subsequently discussed by T. Hayashi † on the basis of an elegant method due to P. Humbert.‡

The generalization

$$f(x) = \int_c^x (x-t)^{m+\nu} u(t)\, dt, \qquad -1 < \nu < 0, \quad m = 0, 1, 2, \cdots \tag{20}$$

which includes both Hirakawa's and Abel's equations § as special cases is easily solved by fractional operators. This can be written

$$f(x) = \Gamma(m+\nu+1)\ {}_cD_x^{-(m+\nu+1)} u(x).$$

Operating on both sides with the symbol ${}_cD_x^{m+\nu+1}$ we have

$$u(x) = \frac{{}_cD_x^{m+\nu+1} f(x)}{\Gamma(m+\nu+1)},$$

$$= \frac{d^{m+2}}{dx^{m+2}} \int_c^x \frac{(x-t)^{-\nu}}{\Gamma(m+\nu+1)\,\Gamma(1-\nu)}\ f(t)\, dt. \tag{21}$$

Recalling elementary properties of the gamma function and making use of (6) we can write (21) as

$$u(x) = \frac{1}{\Gamma(m+\nu+1)\,\Gamma(1-\nu)} \{(-1)^{m+1} \frac{\Gamma(m+\nu+1)}{\Gamma(\nu)} \frac{f(c)}{(x-c)^{m+\nu+1}}$$

$$+ \cdots + (-1)^m \frac{\Gamma(m+\nu)}{\Gamma(\nu)} \frac{f'(c)}{(x-c)^{m+\nu}}$$

$$+ \cdots + \frac{\Gamma(\nu)}{\Gamma(\nu)} \frac{f^{(m+1)}(c)}{(x-c)^{\nu}} + \Gamma(1-\nu)\ {}_cD_x^{-(1-\nu)} f^{(m+2)}(x)\}. \tag{22}$$

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* *The Tôhoku Math. Journal.* Vol. 8 (1915), pp. 38-41.

† *The Tôhoku Math. Journal,* Vol. 10 (1916), pp. 56-59.

‡ *Proc. Edinburgh Math. Soc.,* Vol. 32 (1914), pp. 19-29; Vol. 33 (1915), pp. 35-41.

§ Abel's equation is the special case $m = 0, -1 < \nu < 0$.

But it is clear that if the value of $u(x)$ given by equation (22) is substituted in (20), the integral will not be convergent unless

$$f(c) = f'(c) = \cdots = f^{(m)}(c) = 0.$$

Under these conditions $u(x)$ defined by (22) will be the desired solution. These results can be summarized as follows:

THEOREM 2. *If in equation* (20), $f(x)$ *exists together with its first* $m+2$ *derivatives in the interval* $c \leqq x \leqq x_0$, *and if* $f(c) = f'(c) = \cdots = f^{(m)}(c) = 0$, $f^{(m+1)}(c) \neq 0$, *then the solution of equation* (20) *exists in the open interval* $c < x \leqq x_0$. The solution is given by formula (22).

We can apply these results to equation (19), assuming a to be a positive integer, and obtain the solution with the greatest ease. Since we have ${}_0D_x^{-\frac{1}{2}}\, a(a-1)\, kx^{a-2} = a(a-1)k\, \dfrac{\Gamma(a-1)}{\Gamma(a-\frac{1}{2})} x^{a-3/2}$, it follows that, for $a > 1$,

$$u(x) = k\, \frac{\Gamma(a+1)}{\Gamma(3/2)\, \Gamma(a-\frac{1}{2})}\, x^{a-3/2},$$

and for $a = 1$,

$$u(x) = \frac{k}{\Gamma(3/2)\, \Gamma(\frac{1}{2})}\, \frac{1}{\sqrt{x}} = \frac{2k}{\pi}\, \frac{1}{\sqrt{x}}\,.$$

5. *Lalesco's Integral Equation.* Lalesco's integral equation is a generalization of the problem that we have just solved.*

Thus let us consider

$$f(x) = \int_c^x \frac{G(x,t)}{(x-t)^\nu}\, u(t)\, dt, \tag{23}$$

where we assume that

$$G(x,t) = g_0(t) + g_1(t)\,(x-t) + \cdots + g_n(t)\,(x-t)^n.$$

It is then apparent that (23) can be written in the form

$$f(x) = \Gamma(1-\nu)\, {}_cD_x^{-(1-\nu)}(g_0 u) + \Gamma(2-\nu)\, {}_cD_x^{-(2-\nu)}(g_1 u) \tag{24}$$
$$+ \Gamma(3-\nu)\, {}_cD_x^{-(3-\nu)}(g_2 u) + \cdots + \Gamma(n+1-\nu)\, {}_cD_x^{-(n+1-\nu)}(g_n u).$$

We can now operate on both sides of equation (24) with the operator ${}_cD_x^{n+1-\nu}$ and obtain the following ordinary differential equation of nth order:

$$\Gamma(1-\nu)\, d^n/dx^n\, (g_0 u) + \Gamma(2-\nu)\, d^{n-1}/dx^{n-1}\, (g_1 u) + \cdots$$
$$+ \Gamma(n+1-\nu)\, g_n u = {}_cD_x^{n+1-\nu} f. \tag{25}$$

* T. Lalesco, "Sur l'équation de Volterra," *Journal de Mathématique*, Vol. 73 (1908), pp. 125-202. In particular § 9 of part I.

Since equation (25) is a linear non-homogeneous one, its solution will be given by the method of variation of parameters in the form

$$u(x) = \sum_{i=1}^{n} c_i u_i(x) + \int_c^x W(x,t)\ {}_cD_t^{n+1-\nu} f(t)\ dt, \tag{26}$$

where $u_1(x), \cdots, u_n(x)$ are n linearly independent solutions of the homogeneous equation, $W(x,t)$ is the Cauchy function, and the c_i are arbitrary constants.*

Returning to equation (24) let us operate on both sides with ${}_cD_x^{-\nu}$. We shall then have

$${}_cD_x^{1-\nu} f(x) = \Gamma(1-\nu)\ g_0(x)u(x) + \Gamma(2-\nu)\ {}_cD_x^{-1}\ (g_1 u) + \Gamma(3-\nu)\ {}_cD_x^{-2}\ (g_2 u) + \cdots + \Gamma(n+1-\nu)\ {}_cD_x^{-n}\ (g_n u).$$

Then if ${}_cD_c^{1-\nu} f(x) = 0$, we must have

$$g_0(c)\ u(c) = 0.$$

Operating again with ${}_cD_x^{1} = d/dx$ we get in similar fashion

$$[g_0(x)\ u(x)]'_{x=c} + (1-\nu)\ g_1(c)\ u(c) = 0.$$

Repeating this n times we have as boundary conditions the system:

$$\begin{aligned}
&g_0(c)\ u(c) = 0,\\
&[g_0(x)\ u(x)]'_{x=c} + (1-\nu)\ g_1(c)\ u(c) = 0,\\
&[g_0(x)\ u(x)]''_{x=c} + (1-\nu)\ [g_1 u]'_{x=c} + (1-\nu)(2-\nu)\ g_2(c)\ u(c) = 0,\\
&\quad\cdots\cdots\cdots

$u'(c) = \cdots = u^{(n-1)}(c) = 0$, and it follows that in (26) we get after n successive differentiations

$$\sum_{i=1}^{n} c_i u_i^{(j)}(c) = 0, \qquad (j = 0, 1, \cdots, n-1).$$

Since the determinant of this system is the Wronskian, $W = (u_1, u'_2, \cdots, u_n^{(n-1)})$, which is different from zero for $x = c$, provided $g_0(c) \neq 0$, it follows that $c_1 = c_2 = \cdots = c_n = 0$. *In this case the solution of* (23) *is unique.*

We next turn to the problem of solving (23) when $g_0(x) = g_1(x) = \cdots = g_{r-1}(x) \equiv 0$, $g_r(c) \neq 0$. In this case we replace equation (24) by the following:

$$\begin{aligned} f(x) = {} & \Gamma(r+1-\nu)\, {}_cD_x^{-(r+1-\nu)}[g_r(x)\, u(x)] \\ & + \Gamma(r+2-\nu)\, {}_cD_x^{-(r+2-\nu)}[g_{r+1}(x)\, u(x)] \\ & + \cdots + \Gamma(n+1-\nu)\, {}_cD_x^{-(n+1-\nu)}[g_n(x)\, u(x)], \end{aligned}$$

and equations (27) by the system:

$$\begin{aligned} & g_r(c)\, u(c) = 0, \\ & [g_r(x)\, u(x)]'_{x=c} + (r+1-\nu) g_{r+1}(c)\, u(c) = 0, \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & [g_r(x)\, u(x)]^{(n-r-1)}_{x=c} + (r+1-\nu)[g_{r+1}u]^{(n-r-2)}_{x=c} \\ & \qquad + \cdots + (r+1-\nu)\cdots(n-1-\nu) \\ & \qquad\qquad\qquad\qquad\qquad\qquad g_{n-1}(c)\, u(c) = 0. \end{aligned}$$

By reasoning similar to that in the first case it is clear that equation (23) will have a unique solution if $g_r(c) \neq 0$.

It is next of interest to consider the case where

$$g_0(x) = (x-c)^n G_0(x), \qquad g_1(x) = (x-c)^{n-1} G_1(x), \cdots,$$
$$g_{n-1}(x) = (x-c) G_{n-1}(x).$$

It is further assumed that $G_0(c)$, $G_1(c)$, $\cdots$, $G_{n-1}(c)$, $g_n(c)$ are all different from zero.

Then system (27) is satisfied for any value of $u(c), u'(c), \cdots, u^{(n-1)}(c)$ whatsoever. The solution of (23) is therefore the general one (26) with n arbitrary constants.

Two examples follow:

Example 1. $f(x) = \int_0^x \frac{A_1 t\, u(t)}{(x-t)^{\frac{1}{2}}}\, dt + \int_0^x A_2 (x-t)^{\frac{1}{2}} u(t)\, dt.$

This can be reduced to

$$A_1 \Gamma(\tfrac{1}{2})\, d/dx\, (xu) + A_2 \Gamma(\tfrac{3}{2}) u = {}_0D_x^{3/2} f(x).$$

From $A_1\, d/dx\,(x\bar{u}) + \frac{1}{2}A_2\bar{u} = 0$, we get $\bar{u}(x) = k\,t^c$ where $c = -(A_1 + \frac{1}{2}A_2)/A_1$. Hence we have

$$u(x) = k\,x^c + 1/A_1 \int_0^x (x/t)^c\, {}_0D_t^{3/2} f(t)\, dt,$$

provided the integral exists and $c > -1$.

Example 2. Consider the homogeneous equation

$$A_1 \int_0^x \frac{t^2}{(x-t)^\nu}\ u(t)\, dt + A_2 \int_0^x t\,(x-t)^{1-\nu} u(t)\, dt + A_3 \int_0^x (x-t)^{2-\nu} u(t)\, dt = 0,$$

which reduces to

$$a_0 x^2 u''(x) + a_1 x u'(x) + a_2 u(x) = 0$$

where $a_0 = A_1$, $a_1 = 4A_1 + A_2(1-\nu)$, $a_2 = 2A_1 + A_2(1-\nu) + A_3(2-\nu)(1-\nu)$.

The solution will be

$$u(x) = c_1 x^{r_1} + c_2 x^{r_2},$$

where r_1 and r_2 are roots of the equation

$$a_0 r(r-1) + a_1 r + a_2 = 0.$$

It is easily shown that this function satisfies the original equation provided r_1 and r_2 are greater than -1.

The case of equal roots presents more difficulties. Then the solution will be $u(x) = x^r \log x$ where r is a root of the two equations

$$a_0 r(r-1) + a_1 r + a_2 = 0,$$
$$a_0 r + a_0(r-1) + a_1 = 0.$$

To verify that $u(x)$ is indeed a solution consider the integral

$$I(x) = \int_0^x t^\lambda \log t\, (x-t)^\mu\, dt.$$

Making the transformation $t = xs$, we shall have

$$I(x) = \int_0^1 x^{\lambda+1} s^\lambda\, (\log x + \log s)\, x^\mu\, (1-s)^\mu\, ds =$$

$$x^{\lambda+\mu+1} \left\{ \log x\, \frac{\Gamma(\lambda+1)\,\Gamma(\mu+1)}{\Gamma(\lambda+\mu+2)} + \frac{\Gamma(\lambda+1)\,\Gamma(\mu+1)}{\Gamma(\lambda+\mu+2)} \left[\frac{\Gamma'(\lambda+1)}{\Gamma(\lambda+1)} - \frac{\Gamma'(\lambda+\mu+2)}{\Gamma(\lambda+\mu+2)} \right] \right\}$$

$$= x^{\lambda+\mu+1} \{\log x\, B(\lambda+1,\ \mu+1) + B(\lambda+1,\ \mu+1) [\psi(\lambda+1) - \psi(\lambda+\mu+2)]\}.$$

Substituting $u(x) = x^r \log x$ in the original equation and noting that $A_1B(r+3,\ 1-v) + A_2B(r+2,\ 2-v) + A_3B(r+1,\ 3-v) = 0$, we get for the left hand side the expression

$$\begin{aligned} &A_1B(r+3,\ 1-v)[\psi(r+3) - \psi(r+4-v)] \\ &\quad + A_2B(r+2,\ 2-v)[\psi(r+2) - \psi(r+4-v)] \\ &\qquad\qquad A_3B(r+1,\ 3-v)[\psi(r+1) - \psi(r+4-v)], \end{aligned}$$

which we must show is identically zero.

If we make use of the fact that $\psi(r+2) = \psi(r+1) + 1/(r+1)$, $\psi(r+3) = \psi(r+1) + 1/(r+1) + 1/(r+2)$, it will be seen that the above expression reduces to

$$A_1(r+2) + A_1(r+1) + A_2(v-1).$$

This is readily seen to be equivalent to $a_0 r + a_0(r-1) + a_1$, which is equal to zero.

6. *Sbrana's Integral Equation.* Another type of equation which yields easily to the calculus of fractional operators is that of F. Sbrana who, however, used other methods in his discussion.*

Sbrana's equation is of the form

$$f(x) = \int_0^x u(t)\, K(x-t)\, dt, \tag{28}$$

where
$$K(z) = \int_0^z \frac{e^{cu}}{u^a}\ \frac{e^{c_1(z-u)}}{(z-u)^a}\ du, \qquad 0 < a < 1.$$

If we let $z = v - t$, and $u = x - t$, we shall have

$$K(x-t) = \int_t^x \frac{e^{c(v-t)}}{(v-t)^a}\ \frac{e^{c_1(x-v)}}{(x-v)^a}\ dv.$$

If this is substituted in equation (28) and the order of integration inverted by Dirichlet's formula, we shall have

$$e^{c_1 x} \int_0^x \frac{e^{(c-c_1)v}}{(x-v)^a}\ dv \int_0^v \frac{e^{-ct}u(t)}{(v-t)^a}\ dt = f(x).$$

This equation can be written in the form

$${}_0D_x^{a-1}\{e^{(c-c_1)x}\ {}_0D_x^{a-1}[e^{-cx}\, u(x)]\} = \frac{f(x)\, e^{-c_1 x}}{\Gamma^2(1-a)}\ .$$

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* "Sopra certi equazioni integrali di Volterra, risolubili con procedimenti finiti," *Atti dei Lincei*, Vol. 32 (1) (1923), pp. 156-157.

By successive integrations we obtain the solution

$$u(x) = \frac{e^{cx}}{\Gamma^2(1-a)}\, {}_0D_x^{1-a}\{e^{(c_1-c)x}\, {}_0D_x^{1-a}[f(x)\, e^{-c_1x}]\}.$$

As a special case the kernel can be a Bessel's function since

$$J_0(z) = 1/\pi \int_0^z \frac{e^{-i(z-2u)}}{\sqrt{u}\ \sqrt{z-u}}\, du.*$$

It is easily proved that similar treatment can be made if the kernel is of the general form

$$K(z) = \int_0^z \frac{e^{c_n(z-u_{n-1})}}{(z-u_{n-1})^a}\, du_{n-1} \int_0^{u_{n-1}} \frac{e^{c_{n-1}(u_{n-1}-u_{n-2})}}{(u_{n-1}-u_{n-2})^a}\, du_{n-2} \cdots$$
$$\int_0^{u_1} \frac{e^{cu}\, e^{c_1(u_1-u)}}{u^a\,(u_1-u)^{a_1}}\, du.$$

7. *Applications in Electrical Theory—Heaviside's Calculus.* The following applications to the flow of electricity in a cable are considered both by Heaviside and Carson although the methods which they employ do not explicitly make use of Riemann operators.

Consider the case of the flow of electricity in a non-inductive cable with distributed resistance R and capacity C per unit length subject to an impressed voltage $V_0(t)$ at the point $x = 0$.

The differential equations of the cable are †

$$RI = -(\partial/\partial x)\, V,$$
$$C\,(\partial/\partial t)\, V = -(\partial/\partial x)\, \mathrm{I},$$

where x is the distance measured along the cable from a fixed point ($x = 0$), I is the current at the point x, and V the corresponding potential.

Using a method that is familiar in the theory of linear partial differential equations,‡ we can replace $\partial/\partial t$ by z and by elimination of I obtain the equation

$$z\,R\,C\,V = \partial^2 V/\partial x^2.$$

Symbolically we shall then have

$$V(x,t) = e^{-ax}\, V_1(t) + e^{ax}\, V_2(t),$$

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* Riemann-Weber, *Partiellen Differentialgleichungen*, (Braunschweig), (1900), Vol. 1, § 73; (1919), § 75.

† Starling, *Electricity and Magnetism* (1916), p. 449; Jeans, *Mathematical Theory of Electricity and Magnetism* (1915), p. 333.

‡ Forsyth, *Differential Equations*, 5th ed. (1921), pp. 522-23.

where $a = \sqrt{z\,R\,C}$ and $V_1(t)$ and $V_2(t)$ are arbitrary functions.

By differentiation we get

$$I(x,t) = \sqrt{z\,C/R}\ (e^{-ax}\,V_1 - e^{ax}\,V_2).$$

Assuming that the cable is infinitely long so that the reflected wave is absent we may set $V_2 \equiv 0$ and have as the symbolical solution of our problem

$$I = \sqrt{z\,C/R}\ e^{-ax}\,V_1(t),$$

where $V_1(t)$ is to be determined from the boundary condition. Since this asserts that $V(x,t) \mid_{x=0} = V_0(t)$, it is clear that $V_1(t) \equiv V_0(t)$. Hence the complete solution of the problem is found in the expansion:

$$\begin{aligned} I(x,t) &= \sqrt{z\,C/R}\ e^{-ax}\,V_0(t), \\ &= [\sqrt{C/R}\ \{z^{\frac{1}{2}} + \frac{(Ax)^2}{2!}\,z^{3/2} + \frac{(Ax)^4}{4!}\,z^{5/2} + \cdots\} \\ &\quad - \sqrt{C/R}\ \{z\,Ax + \frac{(Ax)^3}{3!}\,z^2 + \frac{(Ax)^5}{5!}\,z^3 + \cdots\}]\,V_0(t), \end{aligned}$$

where for brevity we have written $A = \sqrt{R\,C}$. Therefore we get

$$I(x,t) = \sqrt{C/R\,\pi}\ \left\{ \frac{d}{dt} + \frac{(Ax)^2}{2!}\,\frac{d^2}{dt^2} + \frac{(Ax)^4}{4!}\,\frac{d^3}{dt^3} + \cdots \right\} \int_0^t \frac{V_0(s)}{(t-s)^{\frac{1}{2}}}\,ds$$

$$-\sqrt{C/R}\ \{(Ax)\,\frac{d}{dt} + \frac{(Ax)^3}{3!}\,\frac{d^2}{dt^2} + \frac{(Ax)^5}{5!}\,\frac{d^3}{dt^3} + \cdots\}\ V_0(t).$$

If $V_0(t) = 1$, that is to say if a unit E. M. F. is impressed at $x = 0$, we get the well-known solution,*

$$I(x,t) = \sqrt{C/R\pi t}\ e^{-\frac{RCx^2}{4t}}.$$

If $V_0(t) = \sin \omega t$, that is to say if an alternating E. M. F. is impressed on the circuit, we have at $x = 0$,

$$\begin{aligned} I(0,t) &= \sqrt{\frac{C}{R\pi}}\ \frac{d}{dt} \int_0^t \frac{\sin \omega s}{(t-s)^{\frac{1}{2}}}\,ds, \\ &= \sqrt{\frac{C}{R\pi}}\ \omega \int_0^t \frac{\cos \omega s}{(t-s)^{\frac{1}{2}}}\,ds. \end{aligned}$$

In establishing the convergence of the series representing $I(x,t)$ in the general case for real values of t greater than 0, it will be sufficient to assume

* Jeans, *loc. cit.*, p. 334.

that $V_0(t)$ is analytic in the neighborhood of $t=0$. Let us first consider $I(x,t)$ as a function of x. If we replace $V_0(s)$ by a series $\sum_{n=0}^{\infty} a_n s^n$ and make the transformation $y=s/(t-s)$, we have

$$A(t) = \int_0^t \frac{V_0(s)}{(t-s)^{\frac{1}{2}}}\, ds = \int_0^\infty \sum_{n=0}^{\infty} t^{n+\frac{1}{2}} a_n \frac{y^n}{(1+y)^{n+\frac{1}{2}}}\, dy,$$

$$= \sqrt{t} \sum_{n=0}^{\infty} \frac{\Gamma(n+1)\sqrt{\pi}}{\Gamma(n+3/2)}\, a_n t^n,$$

$$= \sqrt{t}\, \phi(t),$$

where $\phi(t)$ is an analytic function.

Hence for values of $t \neq 0$, and for n sufficiently large, we have the inequality

$$\left| \frac{d^n}{dt^n} \{\sqrt{t}\, \phi(t)\} \right| < k^n n!, \tag{29}$$

where k is a constant independent of n. From this it follows, making use of Stirling's approximation for $n!$, that

$$\lim_{n=\infty} \left[\frac{|Ax|^{2n-2}}{(2n-2)!} \left| \frac{d^n}{dt^n} \{\sqrt{t}\, \phi(t)\} \right| \right]^{1/n} = 0.$$

Similarly the nth root of the general term of the second series leads to the same limit and the convergence of $I(x,t)$ regarded as a function of x is established. Regarding it as a function of t, we may use

$$\sum_{n=1}^{\infty} \frac{|Ax|^{2n-2}}{(2n+2)!}\, n!\, k^n$$

as a dominating series for the first series and

$$\sum_{n=1}^{\infty} \frac{|Ax|^{2n-1}}{(2n-1)!}\, n!\, k^n$$

as a dominating series for the second.

The case of a cable with distributed resistance R and capacity C with distributed leakage G per unit length leads similarly to the differential equations:

$$RI = -\partial/\partial x\ V,$$

$$(Cz+G)\, V = -\partial/\partial x\ I.$$

If the cable is infinitely long we shall then have

$$V = e^{-\sqrt{R(Cz+G)}\,x}\ V_0(t) =$$

$$\{1 + \frac{x^2(RB)}{2!} + \frac{x^4(RB)^2}{4!} + \frac{x^6(RB)^3}{6!} + \cdots\} V_0(t)$$

$$-\{x\,(RB)^{\frac{1}{2}} + \frac{x^3(RB)^{3/2}}{3!} + \frac{x^5(RB)^{5/2}}{5!} + \cdots\} V_0(t),$$

where we have used the abbreviation $B = Cz + G$.

Making use of the expansion of the operator $(d/dx + k)^n$ given in the second section and writing $\lambda = G/C$, this result may be put into the form,

$$V = e^{-\lambda t}\left[1 + \frac{x^2(RC)}{2!}\ z + \frac{x^4(RC)^2}{4!}\ z^2 + \cdots\right]\{e^{\lambda t}\ V_0(t)\}$$

$$-\ \frac{e^{-\lambda t}}{\sqrt{\pi}}\left[x\,(RC)^{\frac{1}{2}} z + \frac{x^3(RC)^{3/2}}{3!}\ z^2 + \cdots\right]\int_0^t \frac{e^{-\lambda s}\ V_0(s)}{\sqrt{t-s}}\ ds. \tag{30}$$

The solution for the case where unit E. M. F. is impressed at $x = 0$ can be obtained with considerable ease by this formula. We first recall that:

$$\int_0^t \frac{e^{-\lambda s}}{\sqrt{t-s}}\ ds = \int_{-\infty}^t \frac{e^{\lambda s}}{\sqrt{t-s}}\ ds - \int_{-\infty}^0 \frac{e^{\lambda s}}{\sqrt{t-s}}\ ds,$$

$$= \sqrt{\pi/\lambda}\ e^{\lambda t} - \int_{-\infty}^0 \frac{e^{\lambda s}}{\sqrt{t-s}}\ ds.$$

Substituting this value for the integral in (30) and taking the indicated derivatives we get

$$V = e^{-x\sqrt{RG}} + \int_{-\infty}^0 \left[\frac{1}{2}\ \frac{x(RC)^{\frac{1}{2}}}{(t-s)^{3/2}} - \frac{1\cdot 3}{2\cdot 2}\ \frac{x^3(RC)^{3/2}}{3!\,(t-s)^{5/2}}\right.$$

$$\left. + \frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}\ \frac{x^5(RC)^{5/2}}{5!(t-s)^{7/2}} - \cdots\right] \frac{e^{\lambda(s-t)}}{\sqrt{\pi}}\ dt. \tag{31}$$

Making the transformation $t - s = \xi$, (31) reduces to

$$V = e^{-x\sqrt{RG}} + \frac{\sqrt{RC}}{2\sqrt{\pi}} \int_t^\infty \frac{x}{\xi^{3/2}}\left[\ 1 - \frac{(RC)x^2}{4\xi}\right.$$

$$\left. + \frac{(RC)^2x^4}{2!(4\xi)^2} - \frac{(RC)^3x^6}{3!(4\xi)^3} + \cdots\right] e^{-\lambda\xi}\,d\xi$$

$$= e^{-x\sqrt{RG}} + \frac{\sqrt{RC}}{2\sqrt{\pi}} \int_t^\infty \frac{x}{\xi^{3/2}} e^{-(RCx^2/4\xi)}\ e^{-\lambda\xi}\,d\xi.$$

That this, together with the value of $I(x,t)$ derived from it, forms a solution of the original differential equations can be verified by direct substitution. The boundary condition is obviously satisfied.

The general case of the inductive cable can be treated similarly by operational methods, but since it does not involve the use of fractional operators the discussion will be omitted.

A third problem employing fractional derivatives is that of determining the voltage at a cable terminal when an E. M. F., $f(t)$, is impressed on a long cable of distributed resistance R and capacity C per unit length through a condenser of capacity C_0. We have just seen that the current entering a cable whose terminal voltage is V is $\sqrt{C/R}\, z\, V$ and the current flowing into the condenser will be $C_0 z \{f(t) - V\}$ since $f(t) - V$ is the voltage across the condenser.

Equating these two values and dividing by z we obtain the following equation for the determination of V:

$$(C/R\ z^{-\frac{1}{2}} + C_0)\ V = C_0 f(t).$$

The solution is easily found since this is a fractional equation of order $-\frac{1}{2}$. From equation (18) we obtain

$$V(t) = f(0)\ e^{(c/RC_0^2)t} + \int_0^t \{f'(s) - \sqrt{\frac{C}{R\,C_0^2}}\ {}_0D_s^{\frac{1}{2}} f(s)\}\ e^{(c/RC_0^2)\ (t-s)}\ ds.$$

If $f(x) = 1$, this reduces to

$$V(t) = e^{(c/RC_0^2)t} - e^{(c/RC_0^2)t} \sqrt{\frac{C}{RC_0^2\,\pi}} \int_0^t \frac{e^{-(c/RC_0^2)\ s}}{\sqrt{s}}\ ds.$$

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Classification of Quadrics in Hyperbolic Space.

By James Pierpont.

§ 1. *Introduction.*

In this paper,* I wish to present an extremely elementary, though somewhat lengthy method, of classifying quadric surfaces in H_3. Instead of using the usual projective methods of Cayley and Klein, I make use of the more direct methods of Riemann as I have developed in three earlier papers.† A brief resumé of this method now follows.

Let ξ, η, ζ be three real numbers, we say they define a point, whose coördinates are these numbers. How the coördinates are chosen is not specified. The metric is defined by

$$ds = \frac{4R^2 d\sigma}{\lambda} \qquad \lambda > 0, \qquad R > 0$$

where $r^2 = \xi^2 + \eta^2 + \zeta^2$, $\lambda = 4R^2 - r^2$

$$\mu = 4R^2 + r^2, \quad d\sigma = + \sqrt{d\xi^2 + d\eta^2 + d\zeta^2}.$$

H-straights ‡ are defined by

$$\delta \int ds = 0.$$

To perform the variation it is convenient to introduce a new set of coördinates

$$x_1 = \frac{4R^2\xi}{\lambda} \quad x_2 = \frac{4R^2\eta}{\lambda} \quad x_3 = \frac{4R^2\zeta}{\lambda} \qquad x_4 = R\,\mu/\lambda$$

Let us also set

$$[ab] = a_1b_1 + \cdots + a_4b_4 \qquad \{ab\} = a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4.$$

We find the coördinates $x_1 \cdots x_4$ satisfy the relation $\{x^2\} = -R^2$.

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* Cf. P. Barbarin, "Études de Géométrie Analytique Non-Euclidienne," *Mém. Couronnés, Acad. Roy. Bruxelles*, Vol. 60 (1900-1); J. L. Coolidge, *Trans. Amer. Math. Soc.*, Vol. 4 (1903), p. 161; J. T. Bromwich, *Ibid.*, Vol. 6 (1905), p. 275.

† *The Amer. Math. Monthly*, Vol. 30, p. 425; Vol. 31, p. 26 (1923-4); *Proceedings of the Mathematical Congress*, Toronto (1924); *Bull. Amer. Math. Soc.*, 1926. The last two have not yet appeared.

‡ For H-read hyperbolic, for e- read euclidean.

A relation $[ax] = 0$ defines a H-plane; H-straights are the intersection of H-planes and conversely.

In order to have a model of our H-space we will regard ξ, η, ζ as rectangular coördinates of a point in e-geometry as in ordinary analytic geometry. Our H-space in this model is represented by the interior points of the e-sphere $\lambda = 0$. H-planes in the model are e-spheres cutting $\lambda = 0$ orthogonally, and H-straights are e-circles also cutting $\lambda = 0$ orthogonally.

Besides the points so far considered it is necessary to consider the points (ξ, η, ζ) for which $\lambda = 0$. We call these *ideal* and define their x coördinates by

$$x_1 = \xi, \quad x_2 = \eta, \quad x_3 = \zeta, \quad x_4 = 2R;$$

hence for ideal points $\{x^2\} = 0$. We must also introduce certain points called imaginary; their coördinates have the form

$$x_1 = t_1 \cdot i, \quad x_2 = t_2 \cdot i, \quad x_3 = t_4 i, \quad t_4 = t_4 i,$$

where $t_1 \cdots t_4$ are real numbers such that $\{t^2\} = R^2$. Hence imaaginary points also satisfy the relation $\{x^2\} = -R^2$. It is understood that when we speak of points we mean in general non-ideal points.

If a is a point real or imaginary, the locus of the points x such that $\{ax\} = 0$, is called the absolute polar of a. We say x is conjugate to a. If a is imaginary its polar is real; if a is real its polar is imaginary. All H-straights perpendicular to a real plane pass through its pole.

The four planes $x_1 = 0 \cdots x_4 = 0$ form a tetrahedron, the τ tetrahedron. Each face $x_k = 0$ is the polar of the opposite vertex which we call A_k.

Let $a \cdot b$ denote the H-straight joining the points a, b. If x is a point on this straight we denote the length of the segment a, x in the sense a to x by (a, x) etc. Then

$$x_k = \frac{\sinh (x, b)/R}{\sinh (a, b)/R} \cdot a_k + \frac{\sinh (a, x)/R}{\sinh (a, b)/R} \cdot b_k$$

$$= ma_k + nb_k \qquad k = 1 \cdots 4.$$

The cross ratio of four points (a, b, x, x') is $n/m : n'/m'$; it is harmonic when $= -1$. It is important to observe that a bisects the segment x, x', when (a, b, x, x') is harmonic and b is conjugate to a.

The linear transformation

$$x_i = \sum_j a_{ij}x'_j \qquad i, j = 1 \cdots 4, \qquad a_{ij} \text{ real}$$

defines an H-displacement or a motion if $ds = ds'$ and $\{x^2\} = \{x'^2\}$. In particular any e-rotation about A_4 in our model is an H-displacement. The transformation

$$x_1 = x_1', \quad x_2 = x_2', \quad x_3 = x_3' \cosh\theta - x'_4 \sinh\theta,$$
$$x_4 = -x_3' \sinh\theta + x_4' \cosh\theta$$

defines a rotation about the edge $A_1 \cdot A_2$ of the τ tetrahedron.

Finally the angle θ between two curves meeting at a point $(\xi\,\eta\,\zeta)$ is given by

$$\cos\theta = \frac{16R^4}{\lambda^2}\left(\frac{d\xi}{ds}\frac{d\xi'}{ds'} + \frac{d\eta}{ds}\frac{d\eta'}{ds'} + \frac{d\zeta}{ds}\frac{d\zeta}{ds'}\right)$$
$$= \left[\frac{dx}{ds}\,\frac{dx'}{ds'}\right]$$

Hence this angle in H-measure is the same as the angle in e-measure between the corresponding curves in the model.

§ 2. *Classification.*

Any quadratic relation between $x_1 \cdots x_4$ defines a quadric surface. Here we consider only surfaces whose equations are homogeneous, viz.:

$$F = \sum a_{ij}x_ix_j = 0, \qquad a_{ij} = a_{ji}, \qquad i, j = 1 \cdots 4.$$

The a's are real numbers and the determinant $|\,a_{ij}\,| \neq 0$. We call the quadric $x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$ the absolute, it remains invariant under a displacement.

If $a = (a_1 \cdots a_4)$ is a point we call

$$a_1F_1(x) + \cdots + a_4F_4(x) = 0, \qquad F_i = \tfrac{1}{2}\frac{\partial F}{\partial x_i}$$

the polar of a relative to F. Thus the plane $\{ax\} = 0$ which we called the absolute polar of a, is the polar of a relative to the absolute.

Suppose now the point g is such that its polar relative to F coincides with its absolute polar. If g is real, every chord of F through g is bisected by g; that is g is a center of F. If g is an imaginary point, its polar is a

plane of symmetry, i. e. all chords perpendicular to this plane are bisected by it. Such g points must satisfy the equations:

$$\begin{aligned}
(a_{11}-k)g_1 + a_{12}g_2 + a_{13}g_3 + a_{14}g_4 &= 0\\
a_{21}g_1 + (a_{22}-k)g_2 + a_{23}g_3 + a_{24}g_4 &= 0\\
a_{31}g_1 + a_{32}g_2 + (a_{33}-k) + a_{34}g_4 &= 0\\
a_{41}g_1 + a_{41}g_2 + a_{43}g_3 + (a_{44}+k)g_4 &= 0.
\end{aligned} \tag{1}$$

We call these the g-equations. For these to hold it is necessary that k satisfies the equation

$$H = \begin{vmatrix} a_{11}-k & \cdots & a_{14} \\ \cdots & \cdots & \cdots \\ a_{41} & \cdots & a_{44}+k \end{vmatrix} = 0 \tag{2}$$

which we call the H-equation; its roots we denote by a, b, c, d.

If we subject the surface F to a displacement the new surface will have an equation

$$\bar{F} = \sum a'_{ij}x_ix_j = 0$$

whose H-equation will have the same roots as (2). By this displacement the surface F, the point g and its polar will behave as a rigid body. If g is real, imaginary or ideal, it remains such after displacement. This being so, we have only to consider the various possibilities that may arise and in each such case move the surface relative to its center, plane or axis of symmetry in order that its equation takes on a simple form.* The following theorems readily proved except the last, simplify the work:

THEOREM 1. *To two different roots of the H-equation do not correspond the same g.*

THEOREM 2. *If g_a, g_b correspond to two different roots a, b of the H-equation then $\{g_a, g_b\} = 0$.*

THEOREM 3. *Moreover both g_a, g_b cannot be ideal.*

THEOREM 4. *If g_a is real, no g_b can be ideal.*

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* Dr. P. D. Schwartz in a paper given at the May meeting of the American Mathematical Society has shown how conics in H_3 may be classified by this method. Cf. *Bull. Amer. Math. Soc.*, Vol. 32 (1926), p. 315.

THEOREM 5. *The H-equation has at least two real roots.**

I find now the following types of quadric surfaces, the numbers on the right are the types given by Bromwich.

No g necessarily ideal. Roots of H all real.

$\epsilon^2 = 1$. η, arbitrary.

Roots unequal.		$(k-a)(k-b)(k-c)(k-d)=0$	
(1)	g_a real.	$bx_1^2+cx_2^2+dx_3^2-ax_4^2=0$	I
One pair of equal roots		$(k-a)(k-b)(k-c)^2=0$	
(2)	g_c real.	$ax_1^2+bx_2^2+c(x_3^2-x_4^2)=0$	Ib2
(3)	g_c imag., g_a real.	$c(x_1^2+x_3^2)+bx_2^2-ax_4^2=0$	Ib1
Two pairs of equal roots		$(k-a)^2(k-b)^2=0$	
(4)	g_a real.	$b(x_1^2+x_3^2)+a(x_2^2-x_4^2)=0$ or $b(x_1^2+x_2^2)+a(x_3^2-x_4^2)=0$	Id
(5)	g_a, g_b imag.	$a(x_1^2-x_4^2)+b(x_2^2+x_3^2)=0$	
Triple Root		$(k-a)^3(k-b)=0.$	
(6)	g_b real.	$a(x_1^2+x_2^2+x_3^2)-bx_4^2=0$	Ic1
(7)	g_a real.	$bx_1^2+a(x_2^2+x_3^2-x_4^2)=0$	Ic2
(8)	g_a, g_b imag.	$ax_1^2+bx_2^2+2ax_3^2+2\epsilon ax_3x_4=0$	IIIb
Quadruple Root		$(k-a)^4=0$	
(9)	g_a real	$a(x_1^2+x_2^2+x_3^2-x_4^2)=0$	Ie
(10)	g_a necessarily imag.	$a(x_1^2+x_2^2+2x_3^2+2\epsilon x_3x_4)=0$	IIIc
(11)	g_a imag. or ideal. $a(x_1^2+x_2^2)+2ax_3^2+\eta(x_2x_3-x_2x_4)-2ax_3x_4=0$		IVb

* T. J. Bromwich, *Quadratic Forms.* Nr. 3 of the *Cambridge Tracts in Mathematics*, Cambridge, 1906, p. 69. The proof as given by Bromwich rests on the theory of elementary divisors. A very simple proof of a generalization of this theorem which I have found will be given in another paper.

A pair of conjugate imaginary roots.

Two real unequal roots $(k-a)(k-b)(k-\alpha-i\beta)(k-\alpha+i\beta)=0$, $\beta\neq 0$

(12) g_a, g_b imag. $\quad ax_1^2+bx_2^2+2\alpha x_3^2+2\epsilon\sqrt{\alpha^2+\beta^2}\cdot x_3x_4=0$ IIa

(13) $a=b$. g_a imag. $\quad a(x_1^2+x_2^2)+2\alpha x_3^2+2\epsilon\sqrt{\alpha^2+\beta^2}\cdot x_3x_4=0$ IIb

One g necessarily ideal. Roots all real.

One pair of equal roots $\quad (k-a)(k-b)(k-c)^2=0$

(14) g_c ideal, g_a, g_b imag. $\quad ax_1^2+bx_2^2+2cx_3^2+2\epsilon cx_3x_4=0$ IIIa

Two pairs of equal roots $\quad (k-a)^2(k-b)^2=0$

(15) g_a imag., g_b ideal. $\quad a(x_1^2+x_2^2)+2bx_3^2+2\epsilon bx_3x_4=0$ IIId

Triple root. $\quad (k-a)^3(k-b)=0$

(16) g_a ideal, g_b imag.

$$bx_1^2+ax_2^2+2ax_3^2+\eta(x_2x_3-x_2x_4)-2ax_3x_4=0 \qquad \text{IVa}$$

We note that Bromwich has no case corresponding to (5).

We now illustrate on a few cases how the reduction is effected.

Type 6. We move F so that g_a coincides with $A_4=(0,0,0,R)$. The g-equations give

$$a_{41}=a_{42}=a_{43}=0, \qquad a_{44}=-a. \tag{3}$$

We rotate F about A_4 till g_b coincides with $A_1=(iR,0,0,0)$. Then the new coefficients satisfy (3) and in addition the g-equations give

$$a_{11}=b. \quad a_{21}=a_{31}=a_{41}=0. \quad F \text{ now has the form}$$

$$bx_1^2+a_{22}x_2^2+a_{33}x_3^2-ax_4^2+2a_{23}x_2x_3=0.$$

It will not confuse the reader if we omit from the a_{ij} here and in the following the accents. We now rotate F about the edge $A_1\cdot A_4$ of the τ tetrahedron through the angle θ determined by

$$\tan\theta=\frac{a_{23}}{a_{33}-a_{22}}$$

when $a_{22}\neq a_{33}$, in the latter case thru $\theta=45°$. Then F has the form

$$bx_1^2+a_{22}x_2^2+a_{33}x_3^2-ax_4^2=0,$$

whose H-equation is $(k-a)(k-b)(k-a_{22})(k-a_{33})=0$. By hypothesis $H=(k-a)^2(k-b)^2$; hence $a_{22}=a$, $a_{33}=b$ or $a_{22}=b$, $a_{33}=a$. We thus get type 6.

Type 7. We move F so that g_a becomes $A_1=(iR,0,0,0)$. The g-equations give $a_{21}=a_{31}=a_{41}=0$, $a_{11}=a$. We rotate F about A_1 till g_b coincides with $A_2=(0,iR,0,0)$. Then $a_{12}=a_{32}=a_{42}=0$. $a_{22}=b$. It is easy to show that $|2a_{34}|<|a_{33}+a_{44}|$. We can therefore rotate F about the edge $A_1\cdot A_2$ so that $a_{34}=0$ if it is not already 0. F now has the form

$$ax_1^2+bx_2^2+a_{33}x_3^2+a_{44}x_4^2=0,$$

whose H-equation is $(k-a)(k-b)(k-a_{33})(k+a_{44})=0$. Hence either

$$(1^\circ)\quad a_{33}=a,\quad a_{44}=-b;\quad \text{or}\quad (2^\circ)\quad a_{33}=b,\quad a_{44}=-a.$$

The g-equations in case 1° show that g_b may be taken real, which is contrary to hypothesis. Hence only 2° exists and this is type 7.

Type 10. We move F so that g_a coincides with $A_1=(iR,0,0,0)$. The g-equations give $a_{11}=a$, $a_{21}=a_{31}=a_{41}=0$. The plane $x_1=0$ cuts F in the conic Φ whose g-equations are obtained by deleting the first row and column of (1). The H-equation of this system is $H_\Phi=(k-a)^3=0$.

Suppose now that the g-equations of Φ admitted a solution for $k=a$, to which corresponds a real non-ideal point $\gamma=(g_2,g_3,g_4)$, then $g'=(0,g_2,g_3,g_4)$ is a solution of the g-equations of F for $k=a$, and F belongs to type 9. We will suppose therefore that the g-equations of Φ admit only an imaginary γ, then g' is imaginary. We rotate F about A_1 till g' coincides with $A_2=(0,iR,0,0)$, then the g-equations give $a_{12}=a_{32}=a_{42}=0$, $a_{22}=0$ and F has the form

$$ax_1^2+ax_2^2+ax_3^2+\beta x_4^2+2\gamma x_3x_4=0$$

whose H-equation is

$$(k-a)^2\{k^2-(a-\beta)k+\gamma^2-a\beta\}=(k-a)^4=0.$$

Comparing coefficients gives $a-\beta=2a$, $\gamma-a\beta=a^2$; hence $a=a+\epsilon\gamma$, $\beta=\epsilon\gamma-a$, $\epsilon^2=1$, and F has the form

$$a(x_1^2+x_2^2)+(a+\epsilon\gamma)x_3^2-(a-\epsilon\gamma)x_4^2+2\gamma x_3x_4=0.$$

This surface cuts the edge $x_1 = 0, x_2 = 0$ in two points given by

$$(a + \epsilon\gamma)x_3^2 - (a - \epsilon\gamma)x_4^2 + 2\gamma x_3 x_4 = 0,$$

one of which is $(0, 0, 2\epsilon R, 2R)$. If the other is P, we rotate F about $A_1 \cdot A_2$ so that P coincides with $A_4 = (0, 0, 0, R)$ when $a = \epsilon\gamma$. F now has the type (10)

As further illustration of our method let us investigate the cases which arise when

$$H = (k - a)^4 = 0 \qquad \textit{and } g \textit{ is necessarily ideal.} \tag{C}$$

We begin by rotating F about A_4 so that g_a coincides with $L = (0, 0, 2R, 2R)$. The g-equations give now $a_{14} = -a_{13}$, $a_{24} = -a_{23}$, $a_{34} = a - a_{33}$, $a_{44} = a_{33} - 2a$. The corresponding H-equation is

$$(k - a)^2 \begin{vmatrix} a_{11} - k & a_{12} \\ a_{12} & a_{22} - k \end{vmatrix} = (k - a)^4 = 0.$$

Comparing coefficients gives $a_{11} = a_{22} = a$, $a_{33} = \eta$, $a_{13} = \beta$, $a_{23} = \gamma$, hence F has the form

$$a(x_1^2 + x_2^2) + \eta x_3^2 + (\eta - 2a)x_4^2 + 2\beta(x_1 x_3 - x_1 x_4) + 2\gamma(x_2 x_3 - x_2 x_4) + 2(a - \eta)x_3 x_4 = 0 \tag{4}$$

This cuts the edge $x_1 = x_2 = 0$ in L and in another point determined by

$$\eta x_3^2 + (\eta - 2a)x_4^2 + 2(a - \eta)x_3 x_4 = 0.$$

Suppose $(C, 1)$ that $P \neq M = (0, 0, -2R, 2R)$. We rotate F about $A_1 \cdot A_2$ so that P coincides with $A_4 = (0, 0, 0, R)$. These coördinates set in (4) give $\eta = 2a$, and F reduces to the form

$$a(x_1^2 + x_2^2) + 2ax_3^2 + 2\beta(x_1 x_3 - x_1 x_4) + 2\gamma(x_2 x_3 - x_2 x_4) - 2ax_3 x_4 = 0$$

whose first two g-equations for $k = a$ give

$$\beta(g_3 - g_4) = 0, \qquad \gamma(g_3 - g_4) = 0. \tag{5}$$

Suppose $(C, 1, a)$ that β or γ is $\neq 0$, then $g_3 = g_4$. The last two g-equations give

$$\beta g_1 + \gamma g_2 = 0. \tag{6}$$

Thus the g-equations for $k = a$ admit the solution

$$g_a = (\iota\rho\gamma, -\iota\rho\beta, 0, 0) \qquad \rho = \frac{R}{\sqrt{\beta^2 + \gamma^2}}$$

This is contrary to our hypothesis (C) that g_a is necessarily ideal; hence case $(C, 1, a)$ does not exist.

Suppose $(C, 1, b)$ *that* $\beta = \gamma = 0$. The g-equations for $k = a$ admit the solution $g_a = (iR, 0, 0, 0) = A_1$ which is also contrary to (C).

Suppose $(C, 2)$ *that* $P = M$. In this case a rotation about $A_1 \cdot A_2$ does not move M and the foregoing does not apply. We may proceed as follows. Setting the coördinates of M in (6) gives $\eta = a$ and F has the form

$$a(x_1^2 + x_2^2) + ax_3^2 - ax_4^2 + 2\beta(x_1x_3 - x_1x_4) + 2\gamma(x_2x_3 - x_2x_4) = 0$$

whose g-equations for $k = a$ give again (5), (6). Hence if β or γ is $\neq 0$ the g-equations are satisfied by $g_a = (\iota\rho\gamma, -\iota\rho\beta, 0, 0)$, ρ as above, i. e. by a non-ideal point contrary to (C). Finally if $\beta = \gamma = 0$, F reduces to type 9 whose g-equations are satisfied by any g, contrary to (C). Thus case (C) does not exist.

In a similar manner all possible cases may be treated.

YALE UNIVERSITY,
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